The kaobook class

Physics III

Electronics

DC and AC Circuits for Engineers

R.R. Romatoski

2026

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Colophon
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Preface

This textbook covers an introduction to basic electronics for engineers. Its goal is to avoid being an all encompassing reference textbook and instead just focus on the main elements of electronics. It assumes you have taken the calculus based course Physics of Electricity and Magnetism (Physics II) and thus starts with a review of basic laws and concept of electronics. It also covers dc circuit analysis tools, operational amplifiers, transient responses of inductor and capacitor circuits, ac circuit analysis using phasors and impedance, and a brief introduction to frequency response and filters.

Electrical circuits are one kind of system engineers model, analyze, and design. Other systems, like mechanical and thermal, behave similarly as they are fundamentally mathematical models. This textbook is written for all engineers, not specifically electrical engineers. It aims to connect subject matter with other math and engineering topics. For example, linear algebra and first and second order differential equations are referenced. Additionally, some mathematical models of circuits or circuit elements that have analogs to other systems are discussed. Brief references are made directly in the chapter paragraphs, and connections that require more than a few sentences are included after the learning objectives at the end of a chapter.

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The content has been garnered from my experience teaching PHYS306 Electronics for eight years at St. Ambrose University as part of the mechanical engineering curriculum. I was one of the teaching assistants at MIT for Dr. Danagoulian's Course 22.071 Analog Electronics and Analog Instrumentation Design for a year during which Dr. Mitchell taught PHYS306. Each provided excellent notes and experiences as a basis for starting to teach the course myself. Over the years, I have scoured textbooks on the subject and honed the content for the course. This book represents an amalgamation of my efforts on how to present electronics for students.

Dr. Romatoski

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Review of Basic Laws and Concepts of Electronics

DC Circuit Basics 1

This chapter reviews core circuit concepts from the course Physics of Electricity and Magnetism, starting with basic electrical quantities before moving to resistive circuit analysis with Kirchhoff's Laws.

While reading the chapter focus on learning how to

- describe the relationship between charge (q), current (i), voltage (v), and power (p) and energy (w) in words and mathematically.
- draw and apply passive sign convention for voltage polarity and current direction.
- memorize and use Ohm's Law.
- identify and classify circuit elements and sources.
- identify nodes and loops.
- identify circuit elements in parallel and series.
- memorize and apply P=IV.
- memorize and implement Kirchhoff's Current and Voltage Laws (KCL and KVL) also known as the node law and loop law.
- calculate equivalent resistances for series and parallel resistors.
- recognize and write equations for divider circuits.

Finally, this chapter includes an enrichment section introducing the LED (light emitting diode), which is often used to show current is flowing in a circuit.

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1.1 Electrical Quantities

Electric signals are the movement of electric **charges** through a conductor like a wire. Charge, measured in Coulombs (C), is conserved in a circuit. Moving charge over time is **current**

$$i = \frac{dq}{dt} \tag{1.1}$$

which is the flow or rate of change (derivative) of charge q over time t. It is measured in Amperes or Amps (A).

Current is like a vector; it has a magnitude and sign denoting which of two directions current flows. A reference direction for current is arbitrarily chosen, and the sign determines if current flows with the reference direction (positive sign) or opposite it (negative sign).

Remark 1.1.1 Positive charges flow in the positive current direction. Electrons are negative charges so flow in the opposite direction of positive current. Historical reasons led to these definitions.

There are two systems of current—ac and dc. Alternating current (ac) is when charge switches directions periodically, usually as a sine wave. It is ideal for sending charges long distances so is used in power grids and households. Direct current (dc) is when charge moves in only one direction. Small systems that use batteries as a source of power have direct current. These systems have stable constant voltage. Let's stick with dc systems for now.

Remark 1.1.2 Rectifiers can convert from ac to dc systems. Inverters can convert from dc to ac systems.

Charges are moved through work (energy transfer) done by an electric potential difference. The work to move charge is **voltage**

$$v = \frac{dw}{dq} \tag{1.2}$$

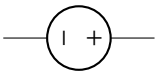

Figure 1.1: Voltage Source Symbol

where w is work in Joules (J), and voltage v has units of Volts (V). Voltage is the potential difference between two points or nodes in a circuit

$$v = \Delta v = v_+ - v_- \tag{1.3}$$

Figure 1.2: Battery Voltage Source Symbol

where voltage has a polarity (two opposite poles) with the higher potential denoted with a plus and lower potential with a minus. Just like with current, a reference polarity is arbitrarily chosen and if the voltage sign is negative then true high and low potential (polarity) are flipped.

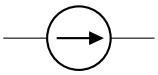

Figure 1.3: Current Source Symbol

A useful analogy for voltage and current is that voltage acts like pressure in a pipe pushing water while current is the resulting flow of water. If two points are labeled A (higher potential/pressure) and B (lower potential/pressure) then $v_{AB} = v_A - v_B = v_+ - v_-$ and the current from point A to B is i_{AB} (assuming it's a single path that does not split).

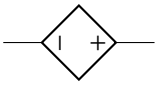

Figure 1.4: Dependent Voltage Source Symbol

Power Sources

Circuits draw power (energy) from voltage and current sources. When diagramming circuits, each circuit element has a specific symbol. Independent voltage source symbols are shown in Figure 1.1 and 1.2, and an independent current source signal is shown in Figure 1.3.

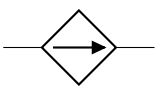

Figure 1.5: Dependent Current Source Symbol

Voltage and current sources can also be dependent sources whose symbols are in Figure 1.4 and 1.5. Dependent sources are also called controlled sources. Each dependent source can be controlled by a current or a

voltage elsewhere in the circuit system. Thus, there can be current controlled voltage sources (CCVS), voltage controlled voltage sources (VCVS), current controlled current sources (CCCS), or voltage controlled current sources (VCCS).

All these sources are treated as ideal and assume no losses. Real world sources have minor non-zero internal resistance; however, assuming ideal sources works really well. Each source has magnitude labeled on the diagram with voltage polarity denoted by plus and minus signs or current direction denoted by an arrow. A dependent source has a magnitude that depends on a voltage or current elsewhere in the circuit which is denoted on the diagram next to the source.

If a source is ac not dc, then a sine wave signal also appears on the source element like shown in Figure 1.6.

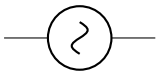

Figure 1.6: AC Source Symbol

1.2 Passive Sign Convention

Every circuit element needs a reference voltage polarity and reference current direction. There is a convention for specifying an element's polarity and current direction called passive sign convention. It requires defining the reference current direction into the positive side of a circuit element as shown in Figure 1.7. Passive sign convention for voltage and current sources are shown in Figure 1.8.

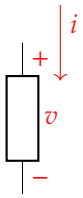

Figure 1.7: Passive Sign Convention Labels

Passive sign convention is critical when analyzing circuits to ensure communication of voltage polarity, current direction, and power.

If values are positive, the voltage polarity and current direction match the diagram. If voltage is negative, the actual voltage polarity is flipped. If current is negative, the actual current direction is the opposite. Negative voltages and currents are not wrong! When labeling polarity and current directions, it is not necessary to pick so that voltage and current are positive because the sign on the value will communicate the actual voltage polarity and current direction.

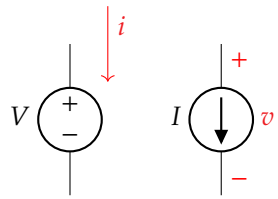

Figure 1.8: Passive Sign Convention of Voltage and Current Sources

If the positive side in Figure 1.7 is labeled A and the negative side B, then

$$v_A - v_B = v_{AB} = -v_{BA} = -(v_B - v_A) \tag{1.4}$$

$$i_{AB} = -i_{BA} \tag{1.5}$$

Circuit analysis tools will not yield correct values if a convention is not followed when voltages and currents get added and subtracted. Therefore, label all elements' voltage polarity and current direction at the start of analysis and do not change them. One crucial law this applies to is Ohm's Law for resistors.

1.3 Resistors

A linear resistor is a two terminal circuit element that opposes the flow of current. It resembles a kink in a water pipe that restricts water flow. Its

Figure 1.9: Resistor Circuit Symbol

circuit symbol is shown in Figure 1.9. Resistors have no defined polarity so may be connected in either direction. They are critical circuit elements for limiting current to ensure it cannot increase high enough to destroy circuit elements. Their resistance R is a measure of opposition to current and is measured in Ohms (Ω).

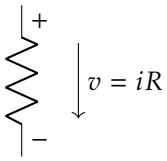

Figure 1.10: Resistor Passive Sign Convention

The elemental relation for resistors with resistance R is governed by Ohm's Law

$$\boxed{v = iR} \tag{1.6}$$

when passive sign convention is followed. Ohm's Law states that voltage across a resistor is directly proportional to the current. This is the current-voltage (i-v) relationship of resistors.

Remark 1.3.1 Polarity or current direction can be defined either way as long as passive sign convention is followed so Ohm's Law applies without thinking about whether a negative sign is needed. If passive sign convention is not followed then Ohm's Law becomes $v = -iR$.

A useful analogy for how Ohm's Law describes how current flows through resistors due to applied voltage is a bucket of water (electric charges) atop a resistor in Figure 1.10 which visually matches Remark 1.7.1 best. The bucket of water is the voltage potential. The water wants to flow down to the ground (negative side of resistor and voltage potential) but the resistor controls the flow rate of water (current) slowing it down.

The opposite of resistance is conductance, which is a measure of ability to conduct rather than restrict current. Conductance is measured in mhos ($\mho$) or Siemens (S). Mathematically it is the reciprocal of resistance so

$$G = \frac{1}{R} = \frac{i}{v} \tag{1.7}$$

where G is conductance and written without fractions $i = Gv$.

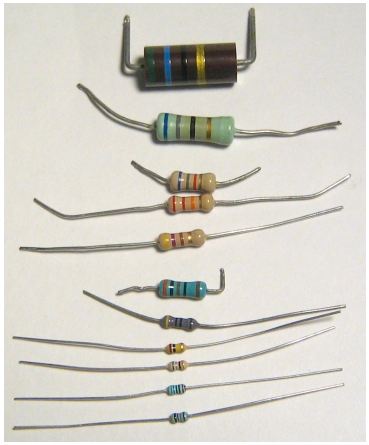

Figure 1.11: Resistors
Omegatron, CC BY-SA 3.0 via Wikimedia Commons

There are fixed resistors with many different constant resistances and variable resistors that allow manual adjustment of the amount of resistance.

1.3.1 Fixed Resistors

For fixed resistors like those shown in Figure 1.11, the color bands on the resistor element indicate their nominal resistance. Actual resistance varies slightly within a specified tolerance and is a source of experimental error that can be measured and quantified. Use Figure 1.15 to read the color bands.

Figure 1.12: Potentiometer Circuit Symbol

1.3.2 Variable Resistors: Potentiometers

A common variable resistor is a potentiometer. These are three terminal not two terminal elements! There are two ways to diagram variable resistors. If the variable resistor is being used as a two terminal resistor, then either Figure 1.13 or 1.12 where one end terminal and the middle terminal are connected may be used.

Figure 1.13: Variable Resistor Circuit Symbol

Alternatively, the potentiometer may be used as a voltage divider using Figure 1.12. The voltage is applied to the two end terminals and the middle terminal is the output voltage that can be varied. It essentially works like two resistors in series (connected in a line), each with varying resistances but always the same total (summed) resistance.

The total resistance is the potentiometer value and its maximum resistance. The minimum resistance is zero so be careful to limit current enough and avoid zero resistance. Best practice is to add a small resistor in series so the resistance along any path in the circuit is never zero.

There are many kinds of potentiometers. One used on breadboards to connect circuit elements is a trim pot shown in Figure 1.14.

Now that we can control current in a circuit with resistors, let's review important terminology and see how to diagram and build circuits with breadboards to connect elements.

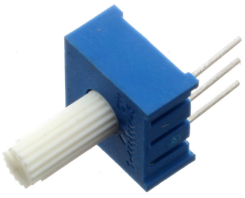

Figure 1.14: Trim Pot
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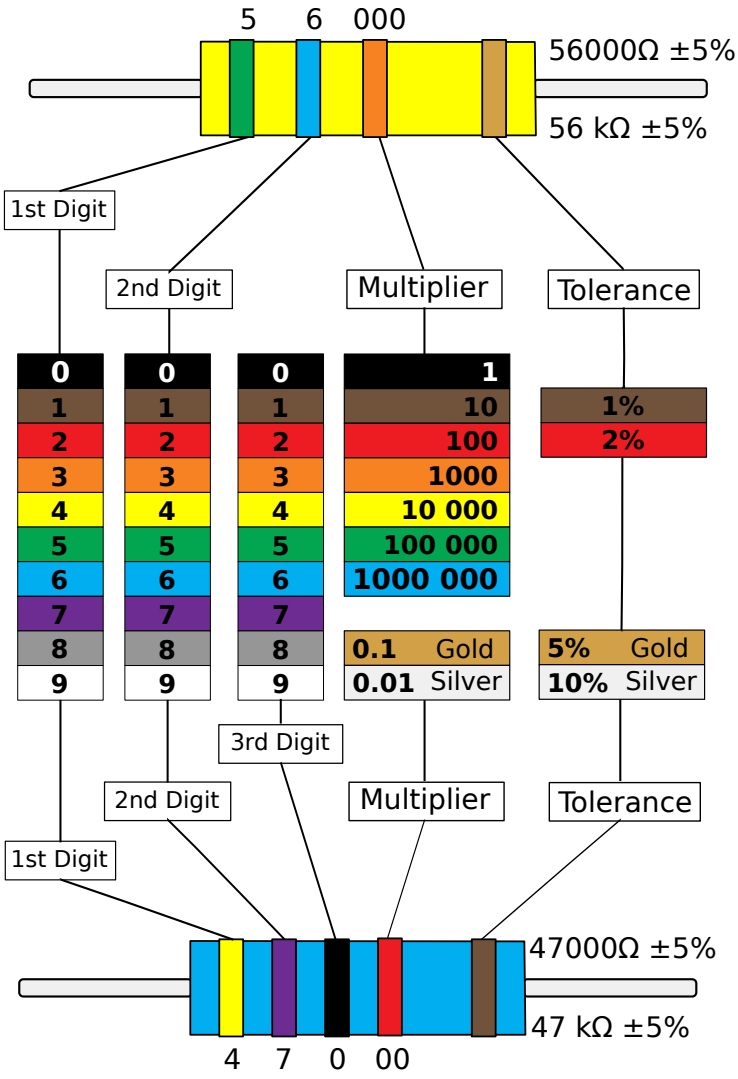


Figure 1.15: Resistor Color Band Decoder
Dirk Koudstaal, CC0 1.0 via Openclipart

1.4 Circuit Diagrams and Terminology

Circuit symbols are used to show a diagram of a circuit for design, analysis, and construction. An example is shown in Figure 1.16. Circuit elements are commonly drawn vertical or horizontal but the point of the diagram is just to show how circuit element terminals are connected. The terminals of each element are connected with lines on the circuit

diagram to show how to connect the element when building a circuit. The lines can be wires to make the connections between elements or the lines can be ignored connecting elements directly.

A two terminal circuit element is a **branch** or path for current to travel. Figure 1.16 has seven branches, one for each circuit element.

The point where two or more circuit element terminals are connected is called a **node**, often indicated as a dot on the diagram. Nodes may also be circled on diagrams. If nodes are difficult to identify with dots, draw a circle or bubble by tracing around connection lines until back at the start and when a circuit element is reached turn around. Figure 1.16 has five nodes labeled a thru e.

When only two elements connect at a node, they are in **series**. Since current cannot split at the node connecting two elements in series, both elements have the same current. Resistors R_2 and R_3 are in series.

When two elements share the same two nodes as connections on each terminal, those two elements are in **parallel** and have the same voltage. Since voltage is a potential difference between two points and the elements share the same two nodes at each end, the voltage across each element is the same. Resistors R_4 and R_7 are in parallel.

For current to travel through a branch/path, it must be part of a **closed circuit** or **loop**. A loop is any closed path where starting from one node, elements connect to other nodes back to the starting node without passing through any node twice. A loop is independent if it contains at least one branch/path not part of another independent loop. Independent loops give rise to independent sets of equations. There are three independent loops in Figure 1.16. In fact, a circuit network with b branches, n nodes, and l independent loops satisfies the equation

$$b = l + n - 1 \tag{1.8}$$

which holds in Figure 1.16 with $7 = 3 + 5 - 1$.

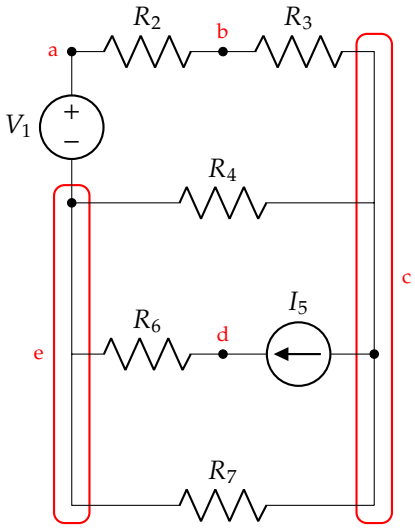


Figure 1.16: Circuit Diagram

A **short circuit** is where there is zero resistance so voltage is zero and current can be any value as governed by Ohm's Law. If a wire connects nodes c and e in Figure 1.16, all the current flows through the wire and none through other branches connecting node c to e. This is called short circuiting. Think of a short circuit as a wire.

An **open circuit** is where resistance approaches infinity. Here the current is zero and voltage can be any value. Think of an open circuit as a break in the circuit, like a wire is cut with a scissors.

To build a circuit, take all the circuit elements (branches) in the diagram and connect them at the connections (nodes) shown in the diagram. One way to connect elements is to use banana plugs and alligator clips; however, this introduces a lot more wires and parts, making it harder to debug (find errors). For prototyping and fabrication, most circuit elements are designed to fit in a breadboard like in Figure 1.17. For finalized designs and products, a printed circuit board (PCB) is often used.

A breadboard has nodes for connecting circuit elements with multiple holes as connection points for each node. Each node is a metal rectangle

as seen on the right of Figure 1.17, which shows the back of a breadboard with the cover removed. Along each vertical side are two nodes labeled with a red plus and blue minus that are called power rails. Typically, sources are connected to these nodes. In the middle, nodes are horizontal rows; however a gap in the middle divides each row into two separate nodes. When building a circuit, if the diagram shows two elements are connected at a node then connect them at a node on the breadboard. The other terminal of each element will be connected to a different node on the breadboard. It may be useful or necessary to use wires on the breadboard to make connections. Once all connections are made on the breadboard, review the diagram and breadboard by checking that connections match for each independent loop.

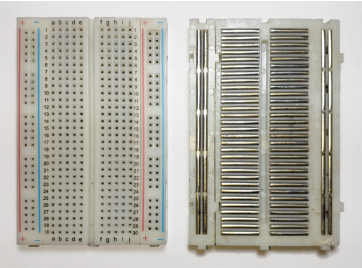


Figure 1.17: Breadboard
Sebastian Wallroth, CC0 1.0 via Wikimedia Commons

1.5 Measuring Electrical Quantities

The two main electrical quantities measured are voltage and current. These can be measured with a multimeter. The multimeter can be treated as a circuit element with the circuit symbol in Figure 1.20. Since circuit elements in parallel have the same voltage, to measure voltage the multimeter should be in parallel with the circuit element. Since circuit elements in series have the same current, to measure current the multimeter should be in series with the desired branch. This requires breaking the circuit and reconnecting it with the multimeter. To help with connecting the multimeter properly, add the meter to the circuit diagram. Incorrect use of a multimeter may blow one of its fuses, which breaks some measuring capabilities until the fuse is replaced. Every multimeter is different and has specifications of what it can measure; however, auto range (Figure 1.18) or manual (Figure 1.19) is one main operational difference.

Figure 1.18: Auto Range Multimeter
Klein Tools, CC BY-SA 4.0 via Wikimedia Commons

Figure 1.19: Manual Range Multimeter
Oomlout, CC BY-SA 2.0 via Wikimedia Commons

1.5.1 Measuring Voltage

Voltage is the potential difference between two points in a circuit. There are two ways to measure voltage between two nodes in a circuit. One way is across an element (or multiple elements). If two elements share nodes on both terminals (are in parallel), then they have the same voltage. Therefore, touch the black lead to the negative side of a circuit element and the red lead to the positive side putting the multimeter in parallel. When recording the voltage of an element, mark the polarity of the leads/element on the diagram so they match and are communicated.

A second way to measure voltage is at a node, which requires a reference node or ground for the potential difference. Typically, ground is the negative side of the source. To measure voltage at a node, touch the black lead to ground and the red lead to the desired node voltage. More details on voltage at a node and ground are in Section 2.1.1.

To measure voltage, plug the black cable in the COM socket and the red cable in the socket labeled V for voltage. Turn the dial to V with a straight line over it for dc systems and with a sine wave over it for ac systems. When the multimeter is in voltage mode, the multimeter has very high resistance so almost no current flows through it. If the

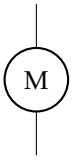


Figure 1.20: Multimeter Circuit Symbol

multimeter requires manually adjusting the range, start by assuming the most dangerous voltage first (highest). If there is no reading, flip to a lower voltage range until there is a measurement reading.

Measuring Resistance

Measuring resistance is similar to measuring voltage except the resistor must have no current. To measure resistance, turn off all sources or remove circuit element. Plug the black cable in the COM socket and the red cable in the socket labeled Ω for resistance. Turn the dial to Ω. Touch the red and black leads each to a different side of the resistor. If the multimeter requires manually adjusting the range, start by assuming the highest range (least sensitive).

1.5.2 Measuring Current

When the multimeter is in current mode, all current from the circuit will flow through the multimeter. So assume the current is high (most dangerous) and in the Amp range. Plug the black cable in the COM socket and the red cable in the socket labeled A for current. Turn the dial to A with a straight line over it for dc and a sine wave over it for ac.

If two elements share a single node (are in series), measure their current by breaking the circuit (think cut with a scissors) at the node and connecting the black and red leads to each side of the break. For measuring current along other branches, break the circuit along the branch/path to be measured and connected the leads to each side of the break. To break the circuit for a desired path, break at one terminal of the circuit element. When recording the current of an element, mark the polarity of the leads so that the current direction is communicated.

If the current is in the mA or smaller range, move the red cable to the socket labeled mA. If the multimeter requires manually adjusting the range, start by assuming the most dangerous current first (highest). If there is no reading, flip to a lower current range until there is a measurement reading. If required, move the red cable to μA socket.

1.6 Power and Energy

In a circuit, the total power supplied is equal to the total power absorbed. Power is the amount of energy transferred or work done over time, which mathematically is

$$P = \frac{dw}{dt} \tag{1.9}$$

and using the relationships between charge, current, and work yields

$$P = \frac{dq}{dt}\frac{dw}{dq}$$

which gives

$$P = iv \tag{1.10}$$

the instantaneous power. Like Ohm's law, Equation 1.10 requires passive sign convention. If power is negative, the circuit element is providing power, and if its positive, it's absorbing (dissipating) power. Resistors always absorb power and dissipate electrical energy as heat.

For each circuit element, power can be calculated, and each power in the circuit summed together will be zero. For resistors, using Ohm's Law yields

$$P = iv = i^2R = \frac{v^2}{R}$$

which means power is not linear even though the current-voltage relationships is linear!

1.7 Kirchhoff's Laws

To theoretically determine voltage and current for all elements in a circuit, Kirchhoff's Voltage and Current Laws (KVL and KCL) can be applied. They come from conservation principles of charge and energy and are the most important laws in electronics as they are the basis of other circuit analysis tools.

1.7.1 Kirchhoff's Current Law (KCL)

Kirchhoff's Current Law (KCL), also known as the node law because it applies at circuit nodes, states that the net current at a node is zero. Conservation of electric charge is the principle behind KCL. Mathematically, KCL is written as

$$\sum_{n=1}^{N} i_n = 0 \tag{1.11}$$

where n is the nth branch at a node and N is the total number of branches connected to the node. The sign of each current is based on its reference direction relative to the node. Follow passive sign convention for reference current direction. It is recommended to use current out of the node as positive to match the convention used later in nodal analysis.

To avoid signs, it is often more intuitive to apply KCL as the current into a node is equal to the current out of a node, which mathematically is

$$\sum i_{in} = \sum i_{out} \tag{1.12}$$

In Figure 1.21, at the top node KCL is $i_s + i_R = 0$ since both currents go out of the node. In Figure 1.22, KCL yields $-I_s + i_R = 0$ or $I_s = i_R$ since I_s goes into the node and i_R goes out.

1.7.2 Kirchhoff's Voltage Law (KVL)

Kirchhoff's Voltage Law (KVL), also known as the loop law because it applies to loops in a circuit, states that the sum of voltages around a closed loop is zero. Conservation of energy is the principle behind KVL. Mathematically, KVL is written as

$$\sum_{m=1}^{M} v_m = 0 \tag{1.13}$$

where m is the mth element voltage or branch in a loop and M is the total number of element voltages or branches around a closed loop. Choose a direction to travel around the closed loop and the sign of each element's voltage depends on its reference polarity determined by passive sign convention. Pick either the sign encountered first or last for each branch (do not mix and match). Follow passive sign convention to use KVL with other analysis tools including KCL and Ohm's Law.

In Figure 1.21, going clockwise around the loop, KVL gives $-V_s + v_R = 0$ using the sign encountered first and $V_s - v_R = 0$ using the sign encountered last, which are equivalent.

1.7.3 Application of Kirchhoff's Laws

Applying KCL to each node and each independent loop results in a set of equations relating current and voltage. There will be more unknowns than knowns. To bridge the gap, use the elemental current-voltage relationships for each circuit element. For resistors, this is Ohm's Law, but note power sources do not have current-voltage relationships. Then solve for the unknowns using substitution or linear algebra (matrices).

Dependent Sources with KCL/KVL

If there are any dependent sources, the dependent source relationship given provides an additional equation for the additional unknown source voltage or current.

The simplest yet most misleading circuit to analyze is a single voltage source or current source with a single resistor. In Figure 1.21, the applied voltage potential V_s sits atop the resistor with resistance R and pushes current down through it (see Remark 1.7.1). To determine how much current is pushed through the resistor, apply Kirchhoff's and Ohm's Laws. First label voltage polarity and current direction using passive sign convention like shown in red in Figure 1.21. Since there is a single loop and only two elements are connected, they are in series; however, each element's reference current goes in different directions. Applying KCL at either node yields

$$i_s + i_R = 0 \rightarrow i_s = -i_R$$

so the currents have the same magnitude but as drawn with passive sign convention they have different signs that communicate there is one true direction of current. Applying KVL around the only closed loop yields

$$-V_s + v_R = 0 \rightarrow V_s = v_R$$

so the voltages have the same magnitude, which confirms they are in parallel. They even happen to have the same sign due to passive sign convention labeling. But these two equations have three unknowns so another equation is needed (ignore that v_R is now known). The elemental relationship of Ohm's Law for the resistor is the final equation that will determine the current for each element as well as the voltage of the resistor. Combining KVL and Ohm's Law then KCL follows as

$$V_s = v_R = i_R R \rightarrow i_R = \frac{V_s}{R}$$
$$i_s = -i_R = -\frac{V_s}{R}$$

so the true direction of current is clockwise since positive $v_R = V_s$ makes i_s negative and i_R positive.

A common mistake is to assume Ohm's Law determines the voltage across the resistor and/or flip its sign. The resistor voltage magnitude will match the source since they are in parallel proven by KVL not Ohm's Law! Also its sign may not match the source just like the current magnitudes are the same but their signs are not. If the voltage polarity of the resistor flips to make the current reference directions the same, then the voltage signs will be opposite of each other but have the same magnitude since they are still in parallel. This is also apparent in Figure 1.22. Simply using passive sign convention and being consistent with signs for Kirchhoff's Laws will allow solving for voltages and currents in any resistive circuit network.

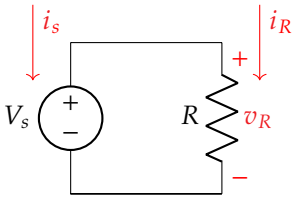
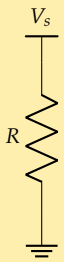


Figure 1.21: Voltage Source and Resistor Circuit

Remark 1.7.1 Another way to draw the circuit in Figure 1.21 is to not explicitly draw the voltage source but instead draw



This also helps illustrate the electric potential as pressure or gravitational potential that pushes current down but is resisted by the resistor on the way to ground.

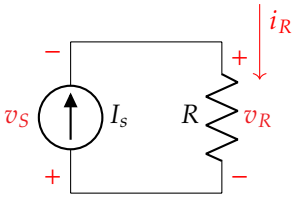


Figure 1.22: Current Source and Resistor Circuit

Since passive sign convention was used, the power for each element in Figure 1.21 is easy to calculate as

$$P_R = i_R v_R = \frac{V_s^2}{R}$$
$$P_s = i_S V_s = -\frac{V_s^2}{R}$$

where the resistor power is positive and absorbs power while the voltage source power is negative and supplies power. Sources do not always supply power unless they are the only source.

1.7.4 Divider Circuits and Equivalent Resistances

The next simplest circuits are current and voltage divider circuits. Analyzing these yield another useful analysis tool for resistors in parallel and series called equivalent resistances.

Voltage Divider and Series Equivalent Resistance

Consider a voltage source in series with two resistors as shown in Figure 1.23 with knowns V_s, R_1, and R_2. First, passive sign convention labels are needed before applying Kirchhoff's Laws as shown in red.

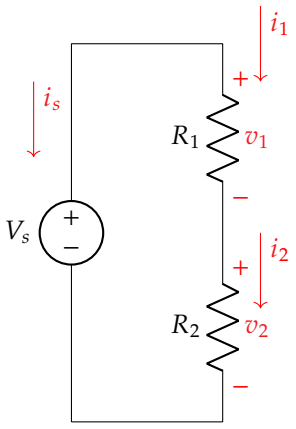


Figure 1.23: Voltage Divider (Series Circuit)

Applying KCL at the top node and the one between resistors yields

$$\begin{aligned} i_s + i_1 &= 0 \quad \rightarrow i_1 = -i_s \\ i_1 = i_2 &= -i_s \end{aligned}$$

where the bottom node equation is redundant. Applying KVL around the only loop yields

$$-V_s + v_1 + v_2 = 0$$

where Ohm's Law for each resistor of $v_1 = i_1 R_1$ and $v_2 = i_2 R_2$ completes the circuits network's set of equations totaling five equations for five unknowns. Solving for the current through the resistors follows as

$$V_s = i_1 R_1 + i_2 R_2 = i_1 (R_1 + R_2)$$
$$i_1 = \frac{V_s}{R_1 + R_2} = i_2$$

which illustrates that if both resistors are combined into an equivalent resistance R_{eq} through addition of resistances then they can be treated as a single resistor. This extends for additional resistors in series. Thus, resistors in series combine as

$$R_{eq,\text{series}} = \sum_{n=1}^{N} R_n \tag{1.14}$$

where n is the nth resistor and N is the total number of resistors in series.

Circling back to calculating voltage for each resistor now that their currents are known yields

$$v_1 = V_s \frac{R_1}{R_1 + R_2}$$
$$v_2 = V_s \frac{R_2}{R_1 + R_2}$$

such that the source voltage is divided between the two resistances. This makes this circuit a voltage divider. In fact, the voltage splits between each resistor as a ratio of its own resistance to their combined resistances. If a voltage divider is recognized, each resistors voltage is known without applying KCL, KVL, and Ohm's Law. However, if not memorized, the voltage divider equations are easily derived.

Current Divider and Parallel Equivalent Resistance

Now consider a current source in parallel with two resistors as shown in Figure 1.24 with knowns I_s, R_1, and R_2. First, passive sign conventions labels are needed before applying Kirchhoff's Laws as shown in red.

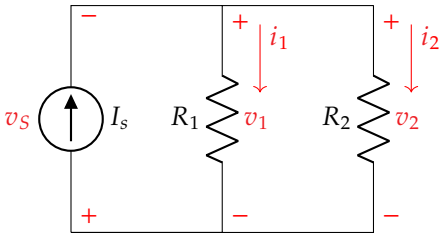


Figure 1.24: Current Divider (Parallel Circuit)

Applying KCL at the top node yields

$$I_s = i_1 + i_2$$

where the bottom node equation is redundant. Applying KVL around each small loop yields

$$\begin{aligned} v_s + v_1 &= 0 \rightarrow v_1 = -v_s \\ -v_1 + v_2 &= 0 \rightarrow v_1 = v_2 = -v_s \end{aligned}$$

where any other loop is redundant (not independent). Lastly applying Ohm's Law for each resistor gives $v_1 = i_1 R_1$ and $v_2 = i_2 R_2$ for a total five equations for five unknowns. Solving for the voltage across the resistors follows as

$$\begin{aligned} I_s &= \frac{v_1}{R_1} + \frac{v_2}{R_2} = v_1 \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \\ v_1 = v_2 &= I_s \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} \end{aligned}$$

which illustrates that if both resistors were combined into $1/R_{eq}$ through addition of inverse resistances then they can be treated as a single resistor. This extends for additional resistors in parallel. Thus, resistors in parallel can be combined as

$$\frac{1}{R_{eq,\text{parallel}}} = \sum_{n=1}^{N} \frac{1}{R_n} \tag{1.15}$$

or

$$R_{eq,\text{parallel}} = \sum_{n=1}^{N} \left(\frac{1}{R_n} \right)^{-1} \tag{1.16}$$

where n is the nth resistor and N is the total number of resistors in series.

For only two resistors, algebra reveals

$$\begin{aligned} v_1 = v_2 &= I_s \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} \\ &= I_s \left(\frac{R_2}{R_1 R_2} + \frac{R_1}{R_1 R_2} \right)^{-1} = \left(\frac{R_2 + R_1}{R_1 R_2} \right)^{-1} \\ &= I_s \frac{R_1 R_2}{R_1 + R_2} \end{aligned}$$

such that two resistors in parallel can be combined as the product over the sum as

$$R_{2eq,\text{parallel}} = \frac{R_1 R_2}{R_1 + R_2} \tag{1.17}$$

which doesn't eliminate fractions but has already combined the terms with a common denominator. Brush up on combining fractions!

Circling back to calculating current for each resistor using Ohm's Law now that their currents are known allows one resistance in the numerator to cancel such that

$$\begin{aligned} i_1 &= I_s \frac{R_2}{R_1 + R_2} \\ i_2 &= I_s \frac{R_1}{R_1 + R_2} \end{aligned}$$

showing that the source current is divided between the two resistances. This makes this circuit a current divider. In fact, the current splits between each resistor (branch) as a ratio of the other resistance to their combined resistances. If a current divider is recognized, each resistors current is known without applying KCL, KVL, and Ohm's Law. However, if not memorized, the current divider equations are easily derived–just be sure to follow the rules for fractions.

1.8 Learning Targets

After reviewing electronics basics from the course Physics of Electricity and Magnetism, you can

- describe the relationship between charge (q), current (i), voltage (v), and power (p) and energy (w) in words and mathematically.
- draw and apply passive sign convention for voltage polarity and current direction.
- memorize and use Ohm's Law.
- identify and classify circuit elements and sources.
- identify nodes and loops.
- identify circuit elements in parallel and series.
- memorize and apply P=IV.
- memorize and implement Kirchhoff's Current and Voltage Laws (KCL and KVL) also known as the node law and loop law.
- calculate equivalent resistances for series and parallel resistors.
- recognize and write equations for divider circuits.

1.9 Diodes

Circuits are very abstract even when built, as the only way to see current and voltage is to measure with a multimeter. However, there is a nice two terminal circuit element that illuminates when current flows through it shown in Figure 1.25 called a light emitting diode (LED). It is a special kind of diode that lights up.

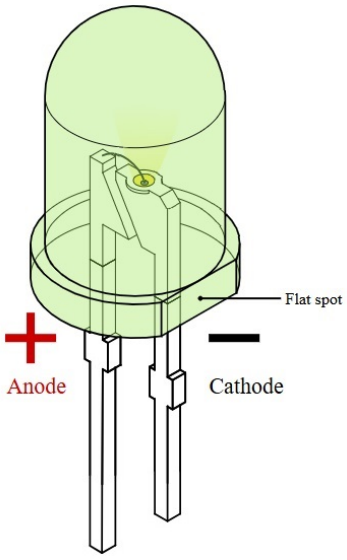


Figure 1.25: LED
Modified from Mrubli, CC BY-SA 4.0 via Wikimedia Commons

This two terminal element with circuit symbol shown in Figure 1.26 is a semiconductor. It has a specific polarity and must be connected in the circuit as shown in the diagram. Its anode is the longer lead and the cathode is the shorter lead. There is a flat spot on the LED's plastic body on the cathode side. The flat spot and the line at the tip of the triangle in the circuit diagram help remind which side of the LED is negative (cathode) as they are like the negative sign shape.

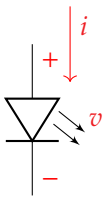


Figure 1.26: LED Circuit Symbol with Passive Sign Convention

A diode allows current to flow only in one direction with minimal leakage in the other direction. For the diode to allow current to flow, enough voltage must be applied. This voltage is called the threshold voltage or forward voltage. Depending on the material used to create the diode, different threshold voltages are required. For silicon, the threshold voltage is about 0.6 V. For LEDs, the different materials emit different wavelengths and thus different colored light. Table 1.1 gives ranges of threshold voltages for different colored LEDs from student experiments for as a rough estimate for different colors. A circuit element's data sheet provides actual specifications. To analyze a circuit with an LED, assume the threshold voltage is the voltage across the LED. If the electric potential at the diode is not at or above the threshold voltage, no current will flow through the diode, and thus if its an LED, no light will be emitted.

Red	1.4 - 1.7
Orange	1.7 - 1.8
Yellow	1.6 - 2.0
Green	1.7 - 2.5
Blue	2.4 - 2.6

Table 1.1: Estimated Range of LED Threshold Voltages

DC Circuit Analysis Tools

Nodal Analysis 2

When the number of equations and unknowns found with Kirchhoff's Laws become unwieldy or resistors cannot be combined, we have two systematic analysis tools–nodal and mesh analysis. Mesh analysis introduces a fictitious loop current and will not be covered. Nodal analysis proves more practical and introduces a new useful way to talk about voltage in a circuit. It also eliminates most labeling with passive sign convention!

While reading the chapter focus on learning how to

- outline the steps of nodal analysis,
- apply them to write a system of equations and solve for unknown node voltages for circuits with
  - current sources
  - one voltage source, and
  - multiple voltage sources (supernodes), and
- use the solved for node voltages to find any current or voltage potential of interest.

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2.1.2 How It Works 20
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2.1 Nodal Analysis Method

2.1.1 Two Ways to Talk About Voltage

Figure 2.1: Two symbols typically used for denoting the ground node

Voltage is a potential difference and thus the voltage difference between two nodes in a circuit. Typically voltage refers to the potential difference **across an element**, which is between two nodes (on either side of the element).

Another way to express voltage is **at a node** in the circuit. But voltage is a potential difference! For voltage at a node, we need another node for reference—each node will use the same one reference or ground node, which has a value of zero volts. Figure 2.1 show the ground circuit symbol.

Ground

The **ground node** is not magic but we assign a reference voltage value to it. The most useful value to assign is zero since it disappears *(magic)* when taking a voltage difference. Thus ground is also referred to as the zero voltage node and reference node. Only one node in a circuit can be designated the ground node.

2.1.2 How It Works

Nodal analysis is just Kirchhoff's Current Law (KCL) applied at each node that creates a solvable system of equations for a circuit network for node voltages when resistor currents are expressed as v/R.

Since each node voltage is in reference to ground, the polarity of each element connected at the node is the positive side. Using passive sign convection, current traveling out of the node is positive. Don't worry this is the only sign convention to remember.

Once the system of equations for the circuit network is solved, those now known node voltages can be used to find any voltage potential or current of interest in the circuit.

The steps of nodal analysis are

1. If desired, combine resistors (in series), select a reference ground node, and assign variable labels to all nodes.
2. Write the circuit network's equations for each node (current out of node is positive) systematically (KCL applied) for each node and supernode. Use KVL for any voltage sources.
3. For any dependent sources, find expressions for controlling variables in terms of node voltages.
4. Put equations into standard form and solve for node voltages.
5. Use known node voltages to calculate any voltage potential or current of interest.

Three examples will demonstrate how to apply these steps for three situations—no voltage sources (current sources only), one voltage source, and multiple voltage sources. No voltage sources will demonstrate the steps to write node equations. The one voltage source will demonstrate how to deal with a voltage source, and the multiple voltage sources will demonstrate how to deal with floating voltage sources not connected to ground.

2.2 No Voltage Sources

Let's see how KCL is applied systematically to write **node equations** without thinking about passive sign convection by applying the nodal anlaylsis steps to the circuit without voltage sources in Figure 2.2 to find the current through resistor R_2.

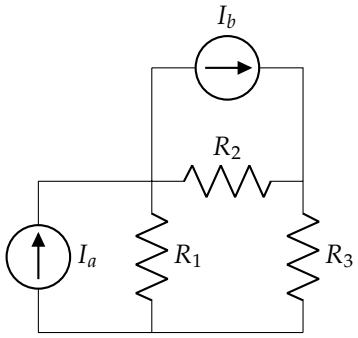

Figure 2.2: No Voltage Source Nodal Analysis Example

1. No series resistors to combine. Without voltage sources, any node can be designated the ground node and labeled e_0; let's pick the bottom node. Label the remaining the nodes with variable e with increasing subscripts as shown in Figure 2.3. The variable v will be reserved for voltage across an element.
2. For each node, apply KCL with unknowns as node voltages as follows:

   (e_0): The ground node voltage is known at zero volts. $e_0 = 0$

   (e_1): Three elements connect to node e_1 so we will sum their three currents to zero to write a **node equation**. Current source I_a goes into the node so needs a negative sign! For the current of a resistor, use Ohm's Law rearranged to $i = v/R$ where v is voltage at the node minus node voltage on the other side of the resistor (voltage across an element). Thus,

   $$-I_a + I_b + \frac{e_1 - e_0}{R_2} = 0$$

   But fractions are ugly and cumbersome! Let's use resistance's reciprocal, conductance G, instead to get

   $$-I_a + I_b + G_2(e_1 - e_0) = 0$$

   (e_2): Repeating from node e_1 for node e_2 gives

   $$-I_b + G_2(e_2 - e_1) + G_3(e_2 - e_0) = 0$$

3. Not applicable
4. Standard form is a matrix form allowing use of linear algebra to solve the system of equations. Standard form groups all unknowns together so their coefficients can be combined, and it puts all known source values on the other side of the equation. Going step by step, first distribute all conductances and substitute for any known voltages like $e_0 = 0$ for ground, keeping node equations in order.

   $$\begin{aligned} -I_a + I_b + G_2 e_1 &= 0 \\ -I_b + G_2 e_2 - G_2 e_1 + G_3 e_2 &= 0 \end{aligned}$$

   Then move all known source values to the other side and put unknown node voltages in order, keeping node equations in order.

   $$\begin{aligned} G_2 e_1 &= I_a - I_b \\ -G_2 e_1 + G_2 e_2 + G_3 e_2 &= I_b \end{aligned}$$

   Next, factor out each unknown node voltage.

   $$\begin{aligned} G_2 e_1 &= I_a - I_b \\ -G_2 e_1 + (G_2 + G_3) e_2 &= I_b \end{aligned}$$

Remark 2.2.1 A common error is not finding all the nodes or finding too many nodes. The voltage does not change at a node. For voltage to change between two points, a circuit element must be in the middle. Use the bubble method from Section 1.4 for locating nodes to help find where voltage changes.

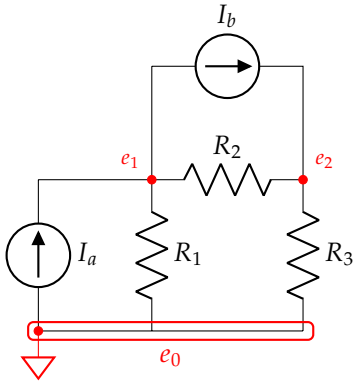

Figure 2.3: No Voltage Source Nodal Analysis Example with Nodes Labeled

Remark 2.2.2 Notice there are three nodes and three equations (including ground node). Also notice the number of terms for each node equation equals the number of elements connected to that node. Each term has units of amps and starts with the node voltage minus another node voltage. Review these to check your work.

Lastly, put in matrix form. If you think about how to multiply the following matrices, putting equations into matrix form is going in reverse.

$$\begin{bmatrix} G_2 & 0 \\ -G_2 & G_2+G_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I_a - I_b \\ I_b \end{bmatrix}$$

The **G** matrix contains known conductances/resistances, the **e** matrix has the unknown node voltages, and the right-hand side matrix or **S** matrix contains known source values.
With only two unknowns, it is easier to just use substitution to solve the two equations for the two unknown node voltages. Cramer's rule from linear algebra solves matrices with more unknowns; however, engineers should solve the matrix equation $\mathbf{e} = \mathbf{G}^{-1} \cdot \mathbf{S}$ using a computational tool like Octave, Matlab, an advanced scientific or graphing calculator, or even Excel.

Remark 2.2.3 Reviewing the matrices will help check your work. Each row corresponds to each node equation for an unknown node voltage. Row 1 is for node e_1 and row 2 is for node e_2. Each column corresponds to each unknown node voltage as listed in the **e** matrix. Notice the diagonal of the **G** matrix is all positive. The conductances summed on the diagonals are for each resistor connected to the node of that matrix row. E.g. the second row is for the e_2 node, so resistors R_2 and R_3 conductances are summed in that second row diagonal element. All off-diagonal matrix elements are negative. The negative conductances summed correspond to the node on the other side of the resistor for each resistor connecting to that row's node. E.g. the second row has $-G_2$ in the first column for the e_2 node row equation because the other side of the R_2 resistor connects to the e_1 node. Some matrix elements are zero because of ground node connections. Lastly, the right-hand side matrix elements are sources, which connect to that row's node. In fact, with enough practice, matrix form can be written directly.

If only two equations and two unknowns, substitution is faster. One can then find that

$$e_1 = \frac{I_a - I_b}{G_2} = R_2(I_a - I_b)$$

and substitute that in to the other equation to find

$$e_2 = \frac{I_b + G_2 e_1}{G_2 + G_3} = \frac{I_b + I_a - I_b}{G_2 + G_3} = \frac{I_b}{\frac{1}{R_2} + \frac{1}{R_3}} = \frac{I_b}{\frac{R_3 + R_2}{R_2 R_3}} = \frac{R_2 R_3 I_b}{R_2 + R_3}$$

5. The goal of nodal analysis is to find currents and voltages of interest. Once all node voltages are found, any current or voltage can be determined. To find the current through resistor R_2, first draw the direction of current with an arrow on the circuit and label the polarity of the resistor. Then use Ohm's law to find the current as

$$i_2 = \frac{v_2}{R_2} = \frac{e_1 - e_2}{R_2}$$

with labeling in Figure 2.4. In finding i_2, v_2 was found also. Any voltage difference or current can be found with the node voltages.

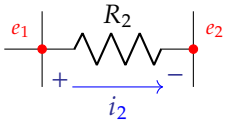

Figure 2.4: No Voltage Source Nodal Analysis Example Resistor R_2 Current Direction and Voltage Polarity Labeled

2.3 One Voltage Source

Let's see what happens with a voltage source and and how to apply KCL at its nodes by applying the steps to the example circuit in Figure 2.5 where the goal is to find the current through resistor R_2.

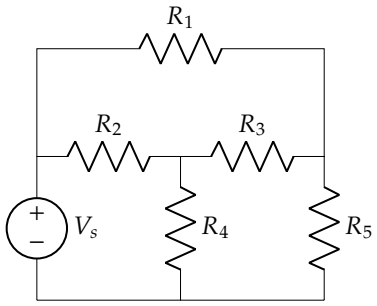

Figure 2.5: One Voltage Source Nodal Analysis Example

1. No series resistors to combine, but there is one voltage source. Thus, the ground node must be the negative side of the voltage source. The reason is forthcoming. Label the remaining nodes as shown in Figure 2.6, which subscript number is where is arbitrary.
2. For each node, apply KCL as follows:

   (e_0): The ground node is zero volts. $e_0 = 0$

   (e_1): Three elements connect to node e_1 but one is a voltage source. What is the current along that path? Ohm's law does not apply. But by definition, the voltage across the voltage source is known! Thus, we can find the node voltage by writing a **source equation** as

   $$e_1 - e_0 = V_s \rightarrow e_1 = V_s$$

   See why ground must be at the negative side of the voltage source; it allows a node voltage be known immediately. Substituting for this known voltage reduces the number of equations and avoids the supernode method needed for floating voltage sources.

   (e_2): Using $i = Gv$ where v is voltage at node minus node voltage on the other side of the resistor, the remaining node equations are found.

   $$G_2(e_2 - e_1) + G_4(e_2 - e_0) + G_3(e_2 - e_3) = 0$$

   (e_3):

   $$G_1(e_3 - e_1) + G_3(e_3 - e_2) + G_5(e_3 - e_0) = 0$$

3. Not applicable
4. Substitute for known node voltages (e_0 and e_1) and rearrange into standard and matrix form.

   $$(G_2 + G_4 + G_3)e_2 - G_2V_s - G_3e_3 = 0$$
   $$(G_1 + G_3 + G_5)e_3 - G_1V_s - G_3e_2 = 0$$

   $$\begin{bmatrix} G_2 + G_4 + G_3 & -G_3 \\ -G_3 & G_1 + G_3 + G_5 \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} G_2V_s \\ G_1V_s \end{bmatrix}$$

   Notice the positive diagonals and negative off-diagonals and that the ride hand side are known currents that depend on our source.
5. To find the current through resistor R_2 using the now known node voltages, first draw the direction of current with an arrow and label the polarity of the resistor using passive sign convention. Then the current through resistor R_2 is

   $$i_2 = \frac{v_2}{R_2} = \frac{e_1 - e_2}{R_2}$$

   using the labeling in Figure 2.7. In finding i_2, v_2 was also found. Any voltage or current of interest can be found with node voltages.

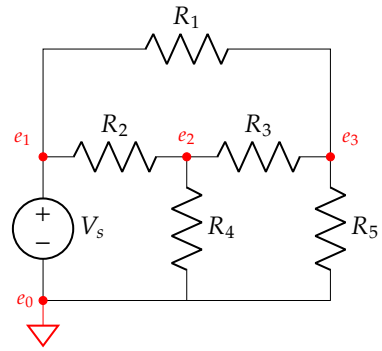

Figure 2.6: One Voltage Source Nodal Analysis Example with Nodes Labels

Remark 2.3.1 Imagine resistor R_1 was initially two resistors in series. It would increase the number of nodes and thus the number of equations for the circuit network. Therefore, combining resistors in series helpfully reduces the number of equations though is not required. Now imagine another resistor is in parallel with resistor R_1 above it. This does not add any nodes! Therefore, ignore resistors in parallel and avoid introducing fractions.

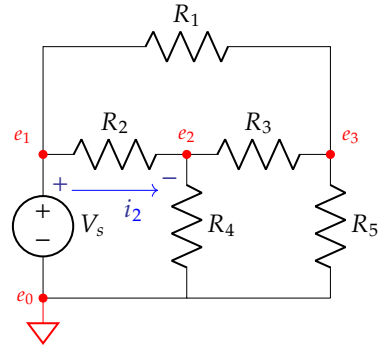

Figure 2.7: One Voltage Source Nodal Analysis Example Resistor R_2 Current Direction and Voltage Polarity Labeled

2.4 Multiple Voltage Sources

Let's see what happens if we have multiple voltage sources without common anodes (negative side) and thus no clear choice for the ground node when applying the steps to the example circuit in Figure 2.8 where the goal is find the current through resistor R_5. Each voltage source could have the ground node. One will be picked and any other "floating" voltage sources will become a **supernode**. If all voltage sources share the same ground node, then each source will provide a known node voltage and is solved as a one voltage source situation.

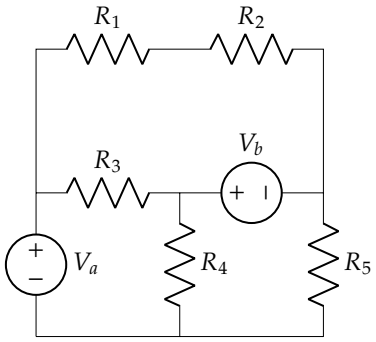

Figure 2.8: Two Voltage Sources Nodal Analysis Example

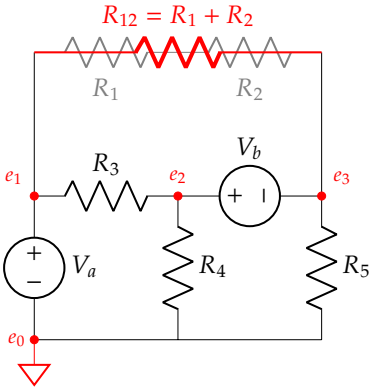

Figure 2.9: Two Voltage Sources Nodal Analysis Example with Node Labels

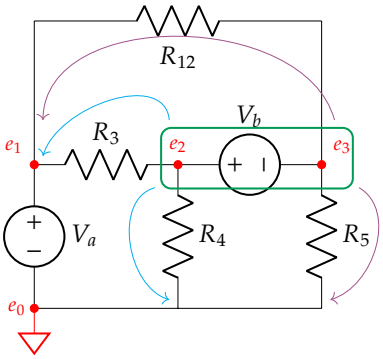

Figure 2.10: Two Voltage Sources Nodal Analysis Example with Supernode

1. Resistors R_1 and R_2 can be combined in series to $R_{12} = R_1 + R_2$. The ground node must be the negative side of a voltage source, but there are two voltage sources with different negative side nodes. Where should ground go? Either negative side works to provide a known node voltage. The other voltage source equation will not provide a known voltage but it will provide another equation. Let's put ground at the negative side of the V_a source. Label the remaining nodes as shown in Figure 2.9.
2. For each node, apply KCL as follows:

   (e_0): The ground node is zero volts. $e_0 = 0$

   (e_1): The voltage source equation yields the node voltage as

   $$e_1 - e_0 = V_a \longrightarrow e_1 = V_a$$

   (e_2): Using $i = Gv$ where v is voltage at node minus node voltage on the other side of the resistor introduces a problem since one path has a voltage source. But we can write a source equation for the voltage source as

   $$e_2 - e_3 = V_b$$

   (e_3): At this point there are three equations but four unknowns. The last equation is a **supernode equation**. Draw a circle around the voltage source and treat it as one big supernode as shown in Figure 2.10. It will have a single supernode equation. However, the supernode equation will have two different voltage at nodes for each side of the voltage source. First write the terms for node e_2 then add terms for node e_3 in one big supernode equation.

   $$G_3(e_2 - e_1) + G_4(e_2 - e_0) + G_{12}(e_3 - e_1) + G_5(e_3 - e_0) = 0$$

   Thus, for the two nodes of the supernode, there are two equations—source equation and supernode equation.

3. Not applicable
4. Substitute for known node voltages (e_0 and e_1). Source equations may be put into matrix form too. Including a source equation or supernode equation will disrupt the typical matrix patterns, but hints of the pattern will emerge especially in larger **G** matrices. Alternatively, substitution for one node voltage before putting into matrix form also works.

$$e_2 - e_3 = V_b$$
$$(G_3 + G_4)e_2 + (G_{12} + G_4)e_3 = G_{12}V_a$$

$$\begin{bmatrix} 1 & -1 \\ G_3 + G_4 & G_{12} + G_4 \end{bmatrix} \begin{bmatrix} e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} V_b \\ G_{12}V_a \end{bmatrix}$$

In this case, substitution works well.

5. To find the current through resistor R_5 using the now known node voltages, first draw the direction of current with an arrow and label the polarity of the resistor using passive sign convention. Then the current through resistor R_5 is

$$i_5 = \frac{v_5}{R_5} = \frac{e_3 - e_0}{R_5} = \frac{e_3}{R_5}$$

using the labeling in Figure 2.11. Any other voltage difference or current can be found with the node voltages.

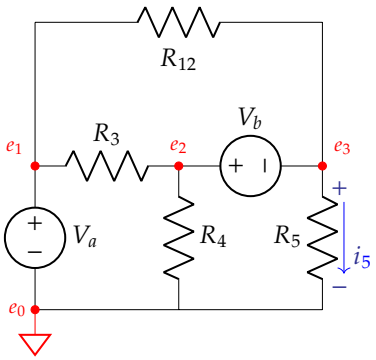

Figure 2.11: Two Voltage Source Nodal Analysis Example with Resistor R_5 Current Direction and Voltage Polarity Labeled

2.5 Learning Targets

Nodal analysis can be added to your circuit analysis toolbox now that you can

- outline the steps of nodal analysis,
- apply them to write a system of equations and solve for unknown node voltages for circuits with
  - current sources
  - one voltage source, and
  - multiple voltage sources (supernodes), and
- use the solved for node voltages to find any current or voltage potential of interest.

Linearity and Superposition 3

If there are numerous sources, the concept of linearity and superposition can be a usefool tool to simplify circuit analysis.

While reading the chapter focus on learning how to

- explain the two properties of linearity and implement them to solve for a circuit output and
- explain the superposition principle and implement it to solve for a circuit output.

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3.1 Linearity

Linearity is a property describing a linear relationship between cause and effect, meaning the output is directly proportional to its input. For a circuit element to be linear, two properties must be satisfied:

Homogeneity property requires that if the input is multiplied by a constant, the output response is multiplied by same constant. Ohm's Law is an example! If a resistor's current is doubled, its voltage doubles too.

Additivity property requires that the output response to the sum of inputs equals the sum of responses to each input applied separately. For example, if a voltage v_1 is applied to a resistor R, its current is $i_1 = v_1/R$, and if an additional voltage v_2 is applied then

$$i = \frac{v_1 + v_2}{R} = \frac{v_1}{R} + \frac{v_2}{R} = i_1 + i_2$$

showing the output equals both the output of the sum of inputs and the output of the sum of each input separately.

So a resistor is a linear element.

Linear circuits consist of linear elements and linear dependent and independent sources.

Power is nonlinear! Though $P = iv$ looks linear, it is not since neither current nor voltage are constants. For a resistor, power is quadratic as

$$P = iv = i^2R = \frac{v^2}{R}$$

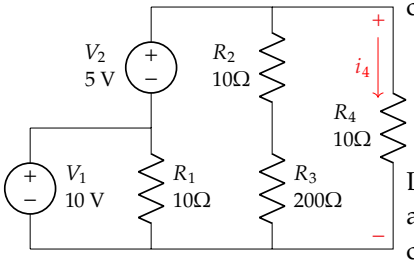

Figure 3.1: Multisource Linear Circuit

Let's examine Figure 3.1 to demonstrate how to use linearity. Using nodal analysis for quick calculations, let's answer three questions about the current i_4 through resistor R_4.

1. **What is the current?** It is 15 V/10 Ω = 1.5 A since the voltage at the positive node of V_2 is 15 V.
2. **If $V_2 = 0$, what is the current?** It is 10 V/10 Ω or 1 A.
3. **If instead $V_1 = 0$, what is the current?** It is 5 V/10Ω or 0.5 A.

The circuit is linear because if the voltage across resistor R_4 doubles from 5 V to 10 V, the current doubles from 0.5 A to 1 A (homogeneity). Additionally, the sum due to each voltage input separately is 1 A + 0.5 A = 1.5 A, which equals the current due to the sum of both voltage sources applied at the same time (additivity). Together this is the superposition principle!

3.2 Superposition

The superposition principle states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of voltages across (or currents through) that element due to each independent source acting alone. Dependent sources must remain because circuit variables control them. The steps to apply superposition are

1. Turn off all independent sources except one and find the desired voltage and/or current due to the remaining active source.
2. Repeat step 1 for each independent source.
3. Find the total contribution by algebraically adding all contributions from each independent source (superpose solutions).

This requires analyzing the circuit multiple times but may reduce the complexity of the analysis.

To turn off a voltage source, voltage needs to be zero. A short circuit has zero voltage potential. To turn off a current source, current needs to be zero. An open circuit has zero current.

Therefore, to turn off (zero) a voltage source change it into a short circuit (wire), and to turn off (zero) a current source change it into an open circuit (break).

This analysis tool is most useful for circuits with few resistors and several sources. After applying superposition to find voltage and current, only then can power be determined.

3.3 Learning Targets

Linearity and superposition can be added to your circuit analysis toolbox now that you can

- explain the two properties of linearity and implement them to solve for a circuit output and
- explain the superposition principle and implement it to solve for a circuit output.

Equivalent Circuits | 4

Thevenin and Norton circuits are two equivalent circuits, meaning their current-voltage (i-v) characteristics are identical. Linear circuits can be simplified to either of these two circuits, enabling straightforward and efficient calculation of the maximum power to a load. Thevenin has a voltage source and Norton has a current source so their equivalency adds source transforms as an analysis tool as well.

While reading the chapter focus on learning how to

- explain in words the Thevenin and Norton Theorems and how to apply them to simplify a circuit,
- determine Thevenin and Norton equivalent circuits and their parameters (V_{th}, I_N, R_{th}, R_N),
- define source transformations and apply them to simplify circuits, and
- calculate maximum power to a load (maximum power transfer).

Finally, this chapter includes an enrichment section on some special equivalent resistive circuits, specifically Wheatstone Bridge and Y/T and Δ/Π conversions.

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4.1 Thevenin and Norton Theorems

A resistor is a linear element. Its current is proportional to voltage with a slope of $1/R$ or G like shown in Figure 4.1. The most general straight line has two parameters and follows the equation of a line

$$y = mx + b \tag{4.1}$$

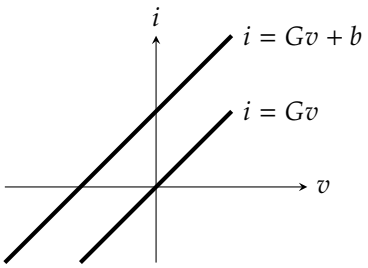


Figure 4.1: Linear Resistor I-V Relationship

where y is the dependent variable, m is the slope, x is the independent variable, and b is a constant. A circuit with this behavior could be

- a resistor in series with a voltage source
- a resistor in parallel with a current source, or
- something more complicated with more circuit elements.

A circuit is linear from the viewpoint of two terminals (nodes) if its i-v relationship is a straight line. Any circuit containing resistors and independent sources is linear like those in Figure 4.2.

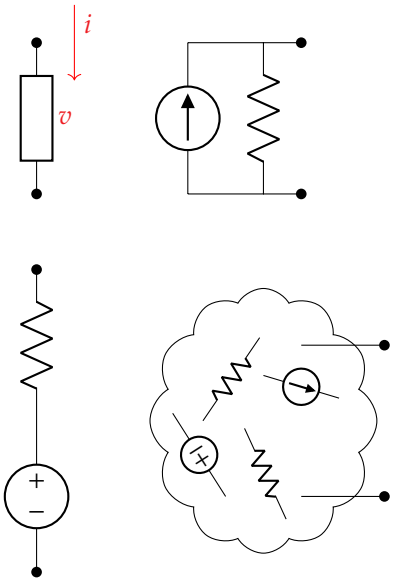
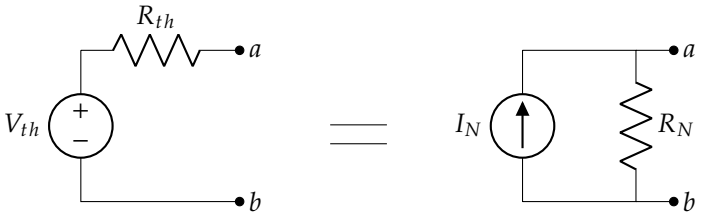


Figure 4.2: Circuits with Linear I-V Behavior

Let's take a "messy" circuit. If the i-v relationship at two terminals let's call a and b is known, then what the circuit consists of does not matter. The circuit can be represented as a voltage source in series with a resistor with the same i-v relationship called a Thevenin equivalent circuit. It could also be represented as a current source in parallel with a resistor with the same i-v relationship called a Norton equivalent circuit.

Thus, **Thevenin's theorem** says any linear circuit of only voltage sources, current sources, and resistors can be replaced at terminals a-b by an equivalent combination of a voltage source V_{th} in series with a resistor R_{th} shown in Figure 4.3a.

Norton's theorem says any linear circuit of only voltage sources, current sources, and resistors can be replaced at terminals a-b by an equivalent combination of a current source I_N in parallel with a resister R_N shown in Figure 4.3b.



(a) Thevenin Equivalent Circuit

(b) Norton Equivalent Circuit

Figure 4.3: Thevenin and Norton Equivalent circuits are source transforms.

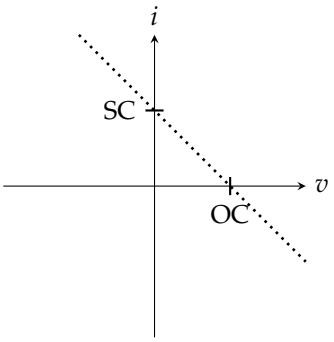


Figure 4.4: Thevenin I-V Relationship

Just like a line needs two parameters or points to define it, two points are needed to find the Thevenin or Norton equivalent circuit at two nodes.

For a Thevenin circuit (Figure 4.3a), two convenient points are when current is zero and voltage is zero (axes intercepts).

- Leaving the circuit open between a-b, current is zero and the voltage across terminals a-b is the open circuit (OC) voltage, which is a point on the v-axis in Figure 4.4. Applying KVL to the Thevenin circuit in Figure 4.3a, the Thevenin voltage [source] is the open circuit voltage across terminals a-b because the resistor voltage is

zero due to no current. Therefore, to find the Thevenin voltage, find the OC voltage across terminals a-b

$$V_{th} = V_{OC} \tag{4.2}$$

with the reference voltage polarity defined as positive at terminal a and negative at terminal b.

- Zero voltage occurs if terminals a-b are a short circuit (SC). This SC current is a point on the i-axis in Figure 4.4. To find the short circuit current for the Thevenin circuit in Figure 4.3a, connect terminals a-b (with a wire). The current equals the resistor current, which is

$$I_{SC} = \frac{V_{th}}{R_{th}} \tag{4.3}$$

with the reference current direction traveling from terminal a to b.

Together these two points describe the $y = mx + b$ line for the i-v relationship of the Thevenin circuit. The Thevenin resistance is then

$$R_{th} = \frac{V_{th}}{I_{SC}} \tag{4.4}$$

There is a method to find the Thevenin resistance without V_{OC} and I_{SC} by finding the equivalent resistance from the viewpoint of terminals a-b with the sources all zeroed. Zeroing a voltage source means making its voltage zero, which occurs if it is a short circuit. To zero a current source, its current must be changed to zero, which occurs if it is an open circuit.

Therefore, to turn off (zero) a voltage source change it into a short circuit (wire), and to turn off (zero) a current source change it into an open circuit (break).

Zeroing the voltage source in the Thevenin equivalent circuit in Figure 4.3a leaves an equivalent resistance of the Thevenin resistance. Therefore, to find the Thevenin resistance, zero the sources and find the equivalent resistance from the viewpoint of terminals a-b.

For a Norton circuit (Figure 4.3b), two points can be found similarly for zero current and zero voltage.

- Leaving the circuit open between a-b, current is zero and the voltage across terminals a-b is the open circuit (OC) voltage. For the Norton circuit in Figure 4.3b, the open circuit voltage is the voltage across the resistor which is

$$V_{OC} = I_N R_N \tag{4.5}$$

with the reference voltage polarity with passive sign convention for the resistor defined as positive at terminal a and negative at terminal.

- Zero voltage occurs if terminals a-b are a short circuit (SC). To find the short circuit current for the Norton circuit in Figure 4.3b, connect terminals a-b (with a wire). The current is the same as that of the current source. Therefore, to find the Norton current, find

the short circuit current across terminals a-b

$$I_{SC} = I_N \tag{4.6}$$

with the reference current direction traveling from terminal a to b.

Together these two points describe the $y = mx + b$ line for the i-v relationship of the Norton circuit. The Norton resistance can then be found as

$$R_N = \frac{V_{oc}}{I_N} \tag{4.7}$$

Alternatively, the Norton resistance can be found by turning off all sources and finding the equivalent resistance from the viewpoint of terminals a-b, which for the Norton circuit is the Norton resistance.

Th/N with Dependent Sources

If there are dependent sources, those sources do not interfere with finding the Thevenin voltage or Norton current; just apply circuit analysis tools to find the open circuit voltage and short circuit current with the dependent source(s). For finding the Thevenin/Norton resistance, however, dependent sources cannot be turned off as they control circuit variables. Instead apply a voltage source at terminals a-b and determine the resulting current or apply a current source at terminals a-b and determine the resulting voltage. The resulting ratio of voltage over current will be the Thevenin/Norton resistance. Any voltage or current source value may be applied. Also note that the resistance may be negative, which indicates the circuit is supplying power.

Let's recap. To find the Thevenin voltage, find the open circuit voltage across terminals a-b. To find the Norton current, find the short circuit current through terminals a-b. To calculate these values for an original circuit, use circuit analysis tools like Kirchoff's Laws, nodal analysis, etc. To find the Thevenin/Norton resistance, zero the sources and find the equivalent resistance from the viewpoint of terminals a-b. Only two of these parameters are needed as the remaining parameter satisfies $V_{OC} = I_{SC}R$ or $V_{th} = I_N R_{th}$.

4.2 Source Transforms

The Thevenin and Norton circuits are equivalent. The same method finds both the Thevenin and Norton resistances so they are equivalent

$$R_{th} = R_N \tag{4.8}$$

The open circuit voltages and short circuit currents of the Thevenin and Norton equivalent circuits are also found using the same approach so

$$V_{OC} = V_{th} = I_N R_N$$
$$I_{SC} = \frac{V_{th}}{R_{th}} = I_N$$

thus the relationship between them all is similar to Ohm's Law

$$V_{th} = I_N R_{th} = I_N R_N \tag{4.9}$$

and the Thevenin and Norton equivalent circuits are equivalent to each other. This makes sense since both theorems take a mess of a circuit and simplify it while keeping the same i-v relationship.

Because Thevenin and Norton circuits are equivalents, a voltage source in series with a resistor in a circuit can be replaced with a current source in parallel with a resistor and visa versa, with the new source magnitude determined by Equation 4.9. Thus, Figure 4.3a equals Figure 4.3b. This is a source transform.

Source Transforms with Dependent Sources

To source transform a dependent source, take care with the algebra for the dependent variable.

Source transforms can be especially useful when resistors are not in series or parallel but the transform rearranges them into series or parallel

resistors. Be careful to maintain the a-b terminal connections when transforming. The voltage source in series with the resistor must not have any other connections between them. It's the connections that matter, not that they are drawn exactly as shown in Figure 4.3.

Figure 4.5 has two possible source transforms. Let's do each one and see they both work. Both Thevenin and Norton equivalent circuits as well as the entire circuit share a-b terminals.

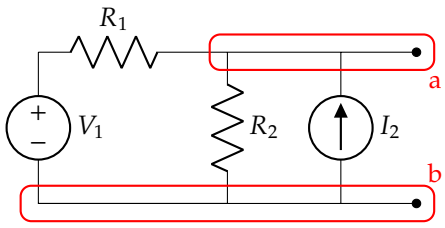


Figure 4.5: Source Transform Example

Source transforming the voltage source The voltage source becomes a current source of V_1/R_1 Amps with the resistors now in parallel as shown in Fig. 4.6. The resistors combine in parallel to $R_{eq} = R_1R_2/(R_1 + R_2)$ and the current sources combine using KCL to $I_{eq} = I_2 + V_1/R_1$ going from node b to a. Only one current source in parallel with one resistor remains, which is a Norton equivalent circuit. So $R_{eq} = R_N$ and $I_{eq} = I_N$. Transforming the Norton equivalent circuit to a Thevenin equivalent circuit yields

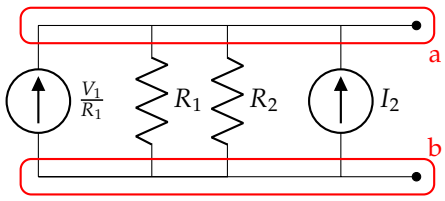


Figure 4.6: Source Transform Example: Transformed Voltage Source

$$R_{th} = R_N = \frac{R_1R_2}{R_1 + R_2}$$

$$V_{th} = I_NR_N = \frac{(I_2 + \frac{V_1}{R_1})R_1R_2}{R_1 + R_2} = \frac{I_2R_1R_2 + V_1R_2}{R_1 + R_2}$$

Source transforming the current source It is tricky to source transform in the middle of a circuit instead of on the edge. In this case it is possible because circling the Norton equivalent circuit and identifying its a-b terminals does not mess up the original circuit a-b terminals as illustrated in Figure 4.7. So transforming yields Figure 4.8. Essentially the current source and parallel resistor can slide to the right edge of the circuit. At this point, one could use the equivalent circuit to find quantities of interest. However, a Thevenin or Norton equivalent circuit cannot be found transforming the current source to a voltage source because after the transform, combining resistors or sources would erase the original a-b terminals.

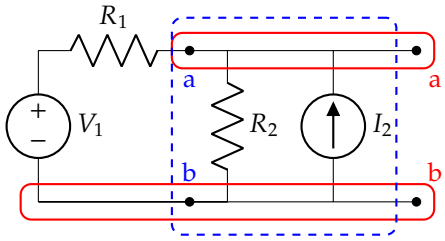


Figure 4.7: Source Transform Example: Transform Current Source

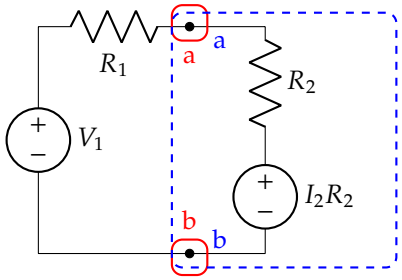


Figure 4.8: Source Transform Example: Transformed Current Source

4.3 Maximum Power Transfer

Circuits can be simplified to analyze powering a load like a heating element or household device. The Thevenin or Norton equivalent circuit with respect to the load is analyzed. To find the maximum power to a load, first determine the equation for the power to the load and then use the first derivative test, i.e. take its derivative with respect to the load resistance and set it equal to zero.

Using the Thevenin equivalent circuit with a load shown in Figure 4.9, the power to the load $P_L = i_Lv_L$ is found as

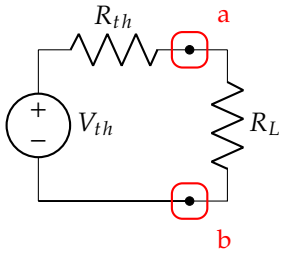


Figure 4.9: Maximum Power Circuit with Thevenin Equivalent

$$v_L = V_{th}\frac{R_L}{R_{th} + R_L} \qquad i_L = \frac{V_{th}}{R_{th} + R_L}$$

$$P_L = \left(\frac{V_{th}}{R_{th} + R_L}\right)^2 R_L$$

where load voltage is found using the voltage divider equation and load current is found by combining resistors in series, using KVL to find

the equivalent resistor voltage, and then Ohm's Law to get the current. Maximum power occurs neither when load resistance is very small or very large. If R_L is zero, power is zero. If R_L approaches infinity, then power goes approaches infinity too. Maximum power is in the middle. The derivative of P_L set to zero is

$$\frac{dP_L}{dR_L} = V_{th}^2 \frac{R_{th} - R_L}{R_{th} + R_L} = 0$$

and thus the maximum power theorem states that the **maximum power delivered to a load occurs when**

$$R_L = R_{th} \tag{4.10}$$

which can then be substituted into the equation for power of the load to find the maximum power. Maximum power is not the most efficient. Efficiency is the ratio of useful power (power of the load) to total power (power of the source) which is

$$\eta = \frac{P_L}{P_{th}} = \frac{I^2 R_L}{I V_{th}} = \frac{I R_L}{I(R_{th} + R_L)} = \frac{R_L}{R_{th} + R_L}$$

which shows that efficiency is low when load resistance is low and highest when load resistance is highest.

4.4 Learning Targets

Thevenin and Norton's Theorems as well as Source Transforms and the Maximum Power Theorem can be added to your circuit analysis toolbox now that you can

- explain in words the Thevenin and Norton Theorems and how to apply them to simplify a circuit,
- determine Thevenin and Norton equivalent circuits and their parameters (V_{th}, I_N, R_{th}, R_N),
- define source transformations and apply them to simplify circuits, and
- calculate maximum power to a load (maximum power transfer).

The electronics toolbox of circuit analysis tools is full now. It includes

- Ohm's Law for resistors $v = iR$
- $P = iv$
- KCL/KVL
- Equivalent resistances
- Nodal analysis
- Linearity/Superposition
- Thevenin's Theorem
- Norton's Theorem
- Source Transforms

With these tools, the puzzle to solve is which to choose and will work best for a given circuit. There are often multiple options, so consider the benefits of each tool—some may be faster, avoid onerous math, or require fewer equations.

4.5 Special Equivalent Resistive Circuits

Sometimes resistors cannot be combing in series or parallel but Thevenin or Norton Theorems or equivalent resistance configuration conversions prove useful.

While reading this section focus on learning how to

- draw circuit diagrams and describe principles of operation for Wheatstone bridge circuits,
- use circuit analysis tools to analyze Wheatstone bridge circuits, and
- apply Y/T and Δ/Π conversions.

4.5.1 Wheatstone Bridge

A Wheatstone bridge circuit is shown in Figure 4.10 where the voltage source may be a resistor depending on the application. One application with the voltage source is determining an unknown resistance of one of the resistors. This setup is useful for measuring strain with a strain guage. To find the unknown resistance, the open circuit voltage across terminals a-b is needed. Let's use Thevenin/Norton to simplify the circuit. The Thevenin resistance is found by zeroing all sources turning the voltage source into a wire and leaving R_1 and R_2 resistors in parallel and R_3 and R_4 resistors in parallel as shown in Figure 4.11. Thus,

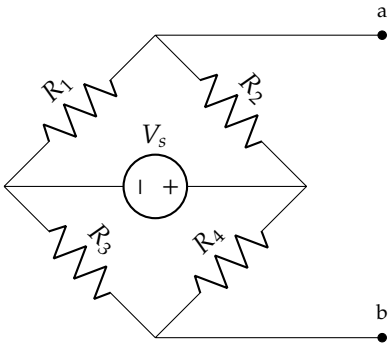


Figure 4.10: Wheatstone Bridge Circuit

$$R_{th} = R_1||R_2 + R_3||R_4 = \frac{R_1R_2}{R_1+R_2} + \frac{R_3R_4}{R_3+R_4}$$

To find the open circuit or Thevenin voltage, a circuit analysis tool will be needed. Since the node voltage at terminal a minus the node voltage at terminal b is the open circuit voltage, nodal analysis is a good tool to apply. Figure 4.12 labels the nodes for analysis.

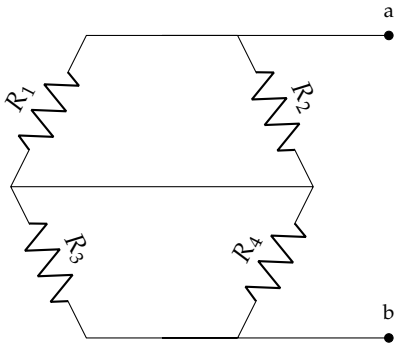


Figure 4.11: Wheatstone Bridge Circuit with Sources Zeroed

Node e_2 is a voltage divider of the voltage source $e_1 = V_s$ and so is node e_3. So the node equations and V_{oc} are

$$\begin{array}{cc} e_0 = 0 & e_1 = V_s \\ e_2 = e_1 \frac{R_1}{R_1+R_2} & e_3 = e_1 \frac{R_3}{R_3+R_4} \end{array}$$

$$V_{oc} = V_{th} = e_2 - e_3 = V_s \left[\frac{R_1}{R_1+R_2} - \frac{R_3}{R_3+R_4} \right]$$

Rearranging V_{th} to include only ratios of resistances yields

$$V_{th} = V_s \left[\frac{\frac{R_1}{R_2}}{\frac{R_1}{R_2}+1} - \frac{\frac{R_3}{R_4}}{\frac{R_3}{R_4}+1} \right]$$

such that if $R_1/R_2 = R_3/R_4$, then $V_{th} = 0$. To take advantage of this, if there's one unknown resistance, say R_4, we can change known resistance R_3 until the circuit is balanced with the open circuit voltage of zero. Then we know the unknown resistance by equating the resistance ratios. Alternatively, we can calculate the unknown resistance with the measured open circuit voltage.

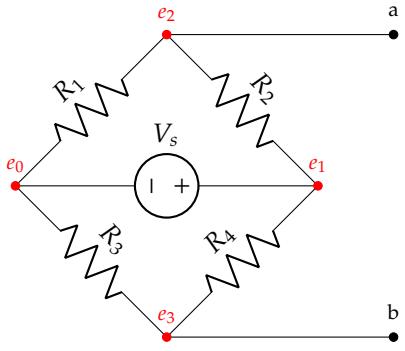


Figure 4.12: Wheatstone Bridge Circuit with Labeled Nodes

4.5.2 Y/T Δ/Π Conversions

Sometimes resisters cannot be combined in series or parallel but connect forming a Y (wye), T (tee), Δ (delta), or Π (pi) shape. Any Y shape can be converted to a Δ shape and any T shape can be converted to a Π shape or visa versa maintaining the node connections as shown in Figure 4.13 and Figure 4.14.

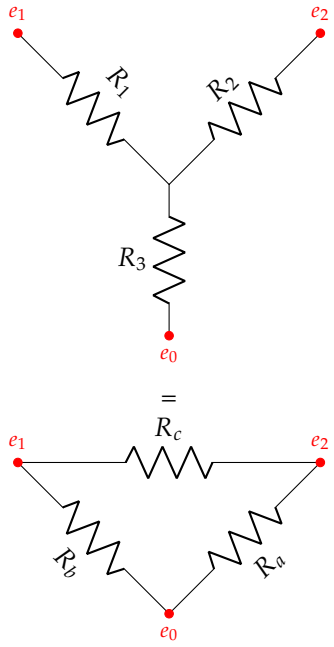


Figure 4.13: Y-Δ Conversions

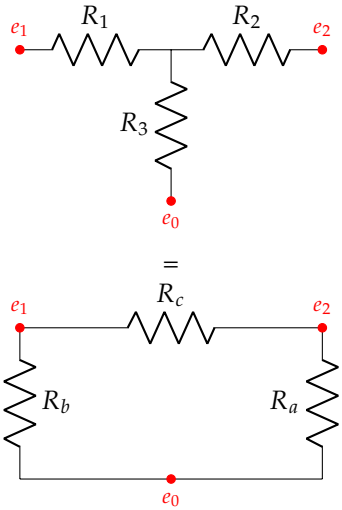


Figure 4.14: T-Π Conversions

These are all equivalent circuits where Y and T have the same connections but are drawn with the middle node below or in line with resistors R_1 and R_2 and where Δ and Π have the same connections but are drawn with resistors directly connected at the bottom node or connected with a wire. Regardless of the shape, the node connections are the same.

The Y and T can be converted to the Δ and Π using

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \tag{4.11}$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \tag{4.12}$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} \tag{4.13}$$

To convert from Δ and Π to Y and T, use

$$R_1 = \frac{R_bR_c}{R_a + R_b + R_c} \tag{4.14}$$

$$R_2 = \frac{R_cR_a}{R_a + R_b + R_c} \tag{4.15}$$

$$R_3 = \frac{R_aR_b}{R_a + R_b + R_c} \tag{4.16}$$

$$\tag{4.17}$$

4.5.3 Special Equivalent Resistive Circuits Learning Targets

With this excursion into special equivalent resistive circuits, you can now

- draw circuit diagrams and describe principles of operation for Wheatstone bridge circuits,
- use circuit analysis tools to analyze Wheatstone bridge circuits, and
- apply Y/T and Δ/Π conversions.

Beyond Resistive Networks

Operational Amplifiers 5

Operational amplifiers (op amps) are neat circuit elements that can perform mathematical operations on circuit signals including addition, subtraction, amplification, and even integration or differentiation of signals.

While reading the chapter focus on learning how to

- describe what op amps can do.
- discuss differences between ideal and nonideal op amps and when to use each model.
- analyze ideal op amps to find input-output relationships.
- analyze cascading op amp circuits.
- understand and interpret nonideal op amps outputs.

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5.1 Op Amp Basics

At its core, an operational amplifier (op amp) amplifies the input signal within the limits of power supplied to the op amp. Op amp components have eight terminals, resembling a spider, as shown in Figure 5.1. To prevent short circuiting connections, the op amp sits in the middle of the breadboard on top of the gap.

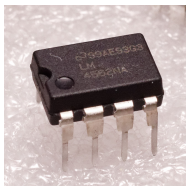

Figure 5.1: Op Amp
Retired electrician, CC0 1.0 via WikiMedia Commons

Figure 5.2 shows the op amp circuit symbol and its connections. Op amps require two power terminals with voltage supply to operate and have three operational terminals, two inputs and one output. From KCL, the op amp output current equals the sum of input currents including those from the two power supplies.

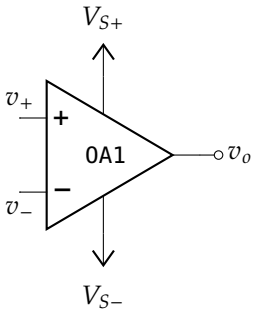

Figure 5.2: Op Amp Symbol

Figure 5.3 shows what each terminal or pin connection is as well as how it corresponds with the circuit symbol. The required connections are

Pin 2: inverting terminal (input inverted [sign flips] at output) v_-
Pin 3: noninverting terminal v_+
Pin 4: negative power supply V_{S-} or V_{EE}
Pin 6: output v_o
Pin 7: positive power supply V_{S+} or V_{CC}

where the power supply is often a symmetric pair of voltages, like ±15 V with an effective ground half way between or the negative supply is ground and the positive supply a positive voltage.

Because a voltage power supply is needed, the op amp is an active circuit element, not passive like resistors.

The op amp acts like a voltage controlled voltage source with an output voltage proportional to the voltage difference of its inputs. Figure 5.4 is an equivalent circuit model of how an op amp works.

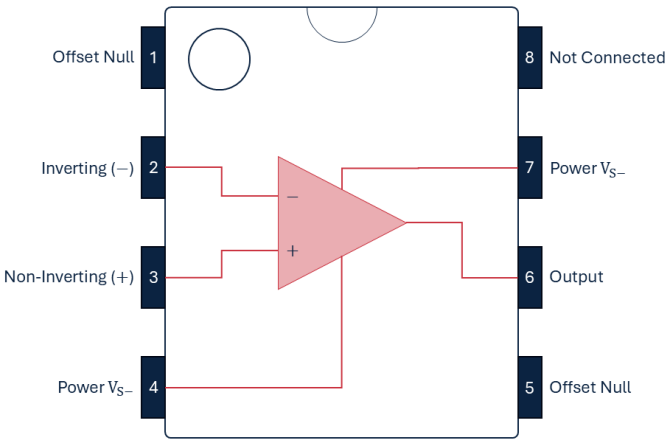

Figure 5.3: Op Amp Connections

The characteristic equation for an op amp is

$$v_o = G(v_+ - v_-) \tag{5.1}$$

where G is the open loop voltage gain, typically on the order of $10^5 - 10^6$. Open loop means there is no feedback from the output to any of the inputs. But feedback is crucial for op amps! The type of feedback or lack thereof determines how to analyze the op amp circuit. There can be no feedback, positive feedback (enhances input signal), or negative feedback (attenuates input signal).

The output voltage cannot exceed the power supply voltages which thereby defines a range for output voltages. When the output reaches the limit of the range it is in saturation. Thus, there are three modes of operation:

1. Positive saturation: output v_o is V_{S+}
2. Linear region: output is between voltage supply range $V_{S-} < v_o < V_{S+}$ which is typically where operation is desired
3. Negative saturation: output v_o is V_{S-}

With no feedback or positive feedback, the op amp operates in saturation only. With negative feedback, the output can be in the linear region where the ratio of output voltage to input voltage, called closed-loop gain, is essentially insensitive to open-loop gain. This is why op amps typically use negative feedback. Additionally, it allows ideal op amp assumptions for analysis.

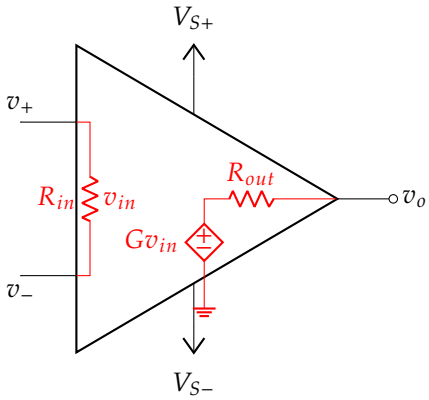

Figure 5.4: Op Amp Equivalent Circuit Model

Remark 5.1.1 In Figure 5.4, notice the input resistance R_{in}, which is low, on the order of 10-100 Ohms, is the Thevenin resistance seen at the input terminals, and the output resistance R_{out}, which is very high, on the order of 10^5-10^{13} Ohms, is the Thevenin resistance seen at the output terminals.

Supply Voltage Labels

This book uses V_{S+} and V_{S-} but some historical labels for supply voltages that are useful to know, specifically from bipolar junction transistors (BJTs) and metal-oxide-semiconductor field-effect-transistors (MOSFETs).

V_{CC} is the positive power supply voltage, specifically voltage at the collector in a BJT

V_{EE} is the negative power supply voltage or ground, specifically voltage at the emitter in a BJT

V_{DD} is the positive power supply voltage, specifically voltage at the drain in a MOSFET

V_{SS} is the negative power supply voltage or ground, specifically voltage at the source in a MOSFET

5.2 Negative Feedback

Working with nonideal op amps with negative feedback is tedious due to their very large open-loop gain. However, treating an op amp as ideal yields essentially the same calculated output. An ideal op amp assumes

- infinite open-loop gain ($G \to \infty$),
- infinite input resistance ($R_{in} \to \infty$), and
- zero output resistance ($R_{out} = 0$)

which means in practice

1. the branch between inputs can be ignored (KCL, nodal analysis) because no current enters the op amp input terminals and
2. $v_+ = v_-$ because the characteristic op amp equation becomes

$$\frac{v_{out}}{G} \to \frac{v_{out}}{\infty} \to 0 = v_+ - v_-$$

Since most real op amps have large gains and input resistance, ideal approximations work well. To see how these assumptions work, the two simplest op amps will be analyzed. Given there are multiple voltage sources, ground as a reference node is very useful and makes nodal analysis (which is KCL at its heart) the leading option to analyze ideal op amps.

5.2.1 Inverting Op Amp

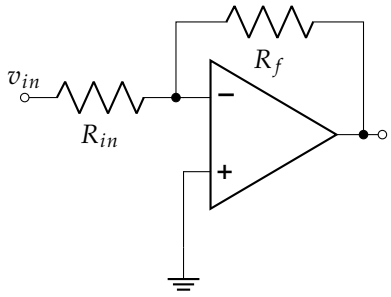

Figure 5.5: Inverting Op Amp Circuit

The inverting op amp is shown in Figure 5.5. It is called inverting because the voltage supply connects to the negative input of the op amp. It is the most basic inverting op amp with one input and one feedback resistor connected to the negative input for negative feedback. To analyze, use nodal analysis and ideal op amp assumptions. At the v_- node, the node equation is

$$\frac{v_- - v_{in}}{R_{in}} + \frac{v_- - v_{out}}{R_f} = 0$$

where v_{out} is the output voltage of the op amp and the ideal op amp assumption has no branch into the op amp. For ideal op amps, $v_+ = v_-$, and $v_+ = 0$ since it is connected to ground. Substituting $v_- = 0$, the node equation simplifies to

$$\frac{-v_{in}}{R_{in}} + \frac{-v_{out}}{R_f} = 0$$

which solved for the ratio of output to input gives

$$\frac{v_{out}}{v_{in}} = -\frac{R_f}{R_{in}} \tag{5.2}$$

where the output to input ratio is defined as the closed-loop gain of the op amp. Notice the voltage sign of the output is opposite the input, confirming this is an inverting op amp.

5.2.2 Non-inverting Amplifier

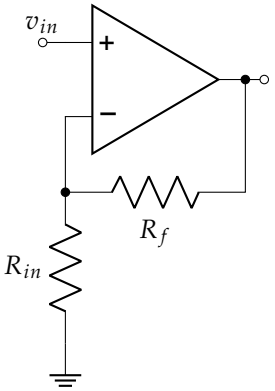

Figure 5.6: Non-Inverting Op Amp Circuit

The noninverting op amp is shown in Figure 5.6. It is noninverting because the voltage supply connects to the positive input of the op amp. It is the most basic inverting op amp because it has one input and one feedback resistor connected to the negative input for negative feedback, just like the inverting op amp. To analyze, use nodal analysis and ideal op amp assumptions. At the v_- node, the node equation is

$$\frac{v_- - 0}{R_{in}} + \frac{v_- - v_{out}}{R_f} = 0$$

where v_{out} is the output voltage of the op amp and the ideal op amp assumption has no branch into the op amp. For ideal op amps, $v_+ = v_-$, and $v_+ = v_{in}$ since it is connected to the input voltage. Substituting $v_- = v_{in}$, the node equation simplifies to

$$\frac{v_{in}}{R_{in}} + \frac{v_{in} - v_{out}}{R_f} = 0$$

which solved for the ratio of output to input is

$$\frac{v_{out}}{R_f} = \frac{v_{in}}{R_{in}} + \frac{v_{in}}{R_f} = v_{in}\left(\frac{1}{R_{in}} + \frac{1}{R_f}\right) \rightarrow \frac{v_{out}}{v_{in}} = R_f\left(\frac{1}{R_{in}} + \frac{1}{R_f}\right)$$

$$\frac{v_{out}}{v_{in}} = 1 + \frac{R_f}{R_{in}} \tag{5.3}$$

where the output to input ratio is defined as its closed-loop gain. The output voltage sign matches the input, confirming it is noninverting.

5.2.3 More Op Amps

In general, op amps with negative feedback can be analyzed as ideal using nodal analysis. Voltage follower, summing, and difference (differential) are some common named op amps. The voltage follower has a gain of one and isolates the impedance of the source and the load to prevent altering the source voltage. A summing op amp adds voltage signals, and a difference op amp subtracts them. Both can be design with a closed-loop gain or to be true summing and difference op amps with a unity gain. They can all be analyzed using nodal analysis.

To illustrate analysis of generic op amps, let's find the output voltage for the op amp example in Figure 5.7. Resistors R_2 and R_3 are in series and can be combined as $R_{23} = R_2 + R_3$ to eliminate one node equation. Node equations for the v_- node and the node between R_1 and R_2 yield

$$\frac{v_- - v_{out}}{R_f} + \frac{v_- - e_1}{R_{23}} = 0$$
$$\frac{e_1 - v_-}{R_{23}} + \frac{e_1 - 0}{R_1} - I_s = 0$$

where e_1 is the node between R_1 and R_2. With the ideal op amp assumption $v_+ = v_-$, $v_- = 0$ thus

$$v_{out} = -\frac{R_f}{R_{23}} e_1$$
$$e_1 \left(\frac{1}{R_{23}} + \frac{1}{R_1} \right) = I_s$$

Solving for e_1 yields

$$e_1 \left(\frac{R_1}{R_{23}R_1} + \frac{R_{23}}{R_{23}R_1} \right) = I_s$$
$$e_1 = I_s \frac{R_1 R_{23}}{R_1 + R_{23}}$$

and then substituting into other equation gives

$$v_{out} = -\frac{R_f}{\cancel{R_{23}}} \frac{I_s R_1 \cancel{R_{23}}}{R_1 + R_{23}}$$
$$v_{out} = -\frac{R_f}{R_1 + R_2 + R_3} I_s R_1$$

An alternative analysis method is to source transform the current source and parallel resistor R_1 then combine all three resistors in series. The simplified circuit is an inverting op amp with feedback resistance R_f, input resistance $R_1 + R_2 + R_3$, and input voltage $I_s R_1$. Then Equation 5.2 also yields the same output voltage.

In analyzing any op amp, if the calculated output voltage exceeds either voltage supply limit, the op amp saturates and the actual output will be the limit it tried to exceed.

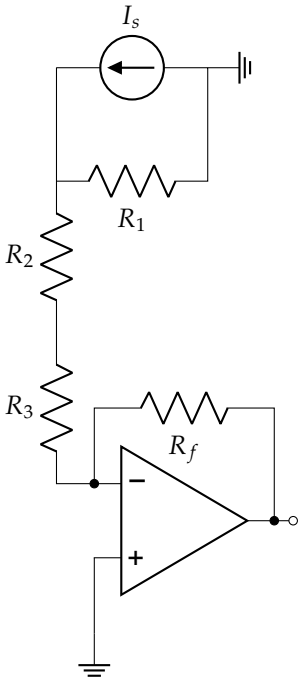

Figure 5.7: Generic Op Amp Circuit

5.2.4 Cascading Op Amps

It is possible and often useful to have multiple op amps in a circuit, which are analyzed with nodal analysis just like a single generic op amp. However, if op amps cascade (connect in series), the output of one op amp becomes the input of the next, with each op amp called a stage.

Cascading op amps can be analyzed with a shortcut where the final stage voltage output is the first stage input voltage multiplied by the product of each stage's closed-loop gain.

For example, consider three cascading stages of an inverting op amp, a noninverting op amp, and lastly another inverting op amp. The output of each stage is

$$\begin{aligned} v_{out,1} &= v_{in,1} \cdot -\frac{R_{f,1}}{R_{in,1}} &&= v_{in,1}A_1 \\ v_{out,2} &= v_{in,2}\left(1 + \frac{R_{f,2}}{R_{in,2}}\right) &&= v_{in,2}A_2 = v_{out,1}A_2 \\ v_{out,3} &= v_{in,3} \cdot -\frac{R_{f,3}}{R_{in,3}} &&= v_{in,3}A_3 = v_{out,2}A_3 \end{aligned}$$

where the output of each op amp in series becomes the input to the next, and each op amp closed-looped gain is A. Thus, the output of the last stage $v_{out,3}$ given the first stage input $v_{in,1}$ is

$$v_{out,3} = v_{in,1}A_1A_2A_3 = v_{in,1} \cdot -\frac{R_{f,1}}{R_{in,1}} \cdot \left(1 + \frac{R_{f,2}}{R_{in,2}}\right) \cdot -\frac{R_{f,3}}{R_{in,3}}$$

yielding a shortcut for op amps cascading (in series) of

$$v_{out,n} = v_{in,1} \prod_{i=1}^{n} A_i \tag{5.4}$$

where i is the ith op amp and n is the last op amp in series.

5.2.5 Calculus with Op Amps

The derivative is a rate of change, while the integral is its opposite. In dc steady state circuits, there is no rate of change. However, capacitor and inductor circuit elements have voltages and currents that change with time allowing op amps to differentiate and integrate input signals as detailed in Section 6.4. Nodal analysis can be used to analyze op amps with capacitors and inductors. However, capacitor and inductor i-v relationships with derivatives and integrals are used in the node equations.

5.3 No Feedback

Without feedback, an op amp is not treated as ideal. Instead, it is analyzed using the characteristic op amp equation (Equation 5.1). Let's look at the output for the op amp in Figure 5.8 for different known input voltages V_+ and V_-.

Case 1: When $V_+ > V_-$, the characteristic equation becomes

$$v_{out} = G \underbrace{(V_+ - V_-)}_{\text{positive value}} \rightarrow v_{out} = V_{S+}$$

because the open loop gain G is very high causing the op amp output to hit the upper saturation limit.

Case 2: When $V_+ = V_-$, the characteristic equation becomes

$$v_{out} = G(V_+ - V_-) = 0$$

Case 3: When $V_+ < V_-$, the characteristic equation becomes

$$v_{out} = \underbrace{(V_+ - V_-)}_{\text{negative value}} \rightarrow v_{out} = V_{S-}$$

because the open loop gain G is very high causing the op amp output to hit the lower saturation limit.

Thus, the op amp in Figure 5.8 will output V_{S+} or V_{S-} depending on which input voltage, V_+ or V_-, is higher. This is called a comparator op amp with one input voltage comparing to a set threshold voltage as the other input operating without feedback.

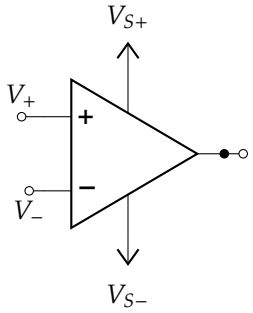

Figure 5.8: Comparator Op Amp Circuit

Comparator Example

If a sinusoid is the V_+ input signal and zero is the set comparator threshold voltage V_-, the output is a square wave that switches to the positive supply voltage when the sinusoid exceeds zero volts and to the negative supply voltage when it falls below zero volts, as shown in Figure 5.9.

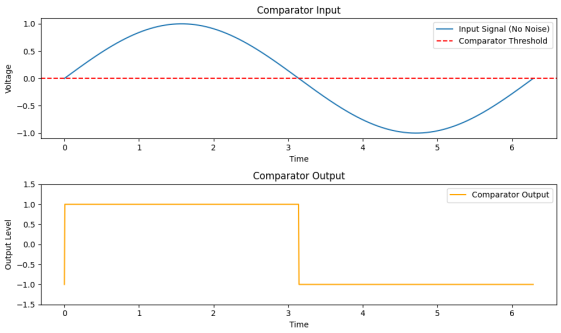

Figure 5.9: Comparator Sinusoidal Input and Corresponding Output Signals Code was generated by ChatGPT (OpenAI) in response to a user query in November 2025. The model produced a Python script for simulating the output of an ideal comparator op-amp driven by a sine-wave input.

If a nonzero threshold voltage is desired, a voltage divider can be used to set the threshold. Additionally, the threshold voltage to compare against may be either the inverting (−) or noninverting (+) input. If the threshhold is the noninverting input then the comparator output in Figure 5.9 would be flipped high to low and low to high.

If the input signal is noisy rather than a nice smooth sinusoid, false threshold crossings due to the noise appear in the output signal square wave, as shown in Figure 5.10. Using an upper and lower threshold instead of one threshhold can eliminate these false crossings.

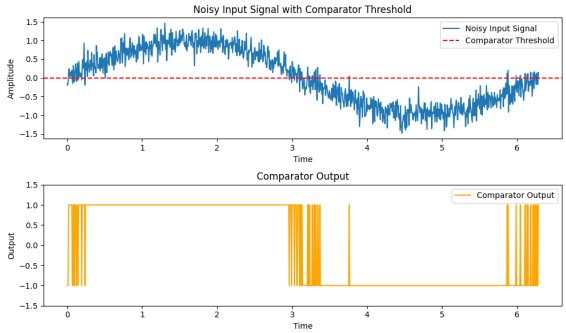

Figure 5.10: Comparator Noisy Sinusoidal Input and Corresponding Output Signals
Code was generated by ChatGPT (OpenAI) in response to a user query in November 2025. The model produced a Python script for simulating the output of an ideal comparator op-amp driven by a noisy sine-wave input.

5.4 Positive Feedback

An op amp with positive feedback will enhance the input signals. Additionally, positive feedback can change the threshold voltage during operation to an upper and a lower threshold. This op amp is called a Schmitt Trigger.

With positive feedback, the op amp is not treated as ideal. Instead, it is analyzed using the characteristic op amp equation (Equation 5.1) to set the comparator thresholds.

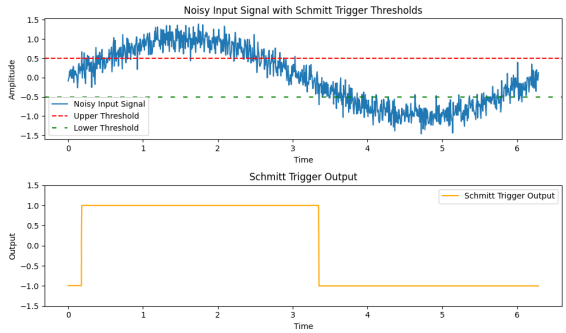

Figure 5.11: Noninverting Schmitt Trigger Noisy Sinusoidal Input and Corresponding Output Signals
Code was generated by ChatGPT (OpenAI) in response to a user query in November 2025. The model produced a Python script for simulating the output of an ideal noninverting Schmitt Trigger op-amp driven by a noisy sine-wave input.

Figure 5.11 shows this behavior for a noninverting op amp with positive feedback, specifically the op amp in Figure 5.12. When the input signal is low, it must rise above the upper threshold to trigger a high output signal. Once the high output signal is triggered, the threshold decreases to the lower threshold. This prevents noise from triggering a low output until signal falls below this lower threshold. The dependence of the output signal response on the input signal's history (starting above or below the threshold) is called hysteresis.

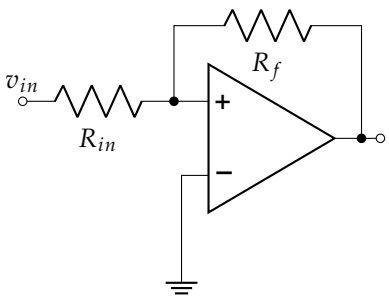

Figure 5.12: Noninverting Schmitt Trigger

Noninverting Schmitt Trigger Example

In Figure 5.12, two thresholds arise when $V_+ = 0$ (intended threshold voltage without feedback) due to two different v_{out} voltages that depend on the output's saturation limit. Using superposition for sources v_{in} and v_{out}, V_+ is

$$V_+ = \frac{R_f}{R_{in} + R_f} v_{in} + \frac{R_{in}}{R_{in} + R_f} v_{out}$$

which when set equal to zero yields

$$\frac{1}{R_{in}+R_f}\left(R_f v_{in}+R_{in}v_{out}\right)=0 \rightarrow v_{in}=-\frac{R_{in}}{R_f}v_{out}$$

where v_{out} is the output signal's history of positive or negative saturation voltages $\pm V_s$. Therefore, the threshold triggers at $\pm V_s$ are when

$$v_{in}=-\frac{R_{in}}{R_f}\cdot \pm V_s$$

where Figure 5.11 shows this behavior for a resistor ratio R_{in}/R_f of 0.5 with supply voltages of ±1 volt, yielding upper and lower threshold voltages of ±0.5 volts.

When v_{in} is higher than 0.5, v_{out} hits positive saturation requiring the input to reach the lower threshold before flipping to positive saturation. When v_{in} is lower than 0.5, v_{out} hits negative saturation requiring the input to reach the higher threshold before flipping to negative saturation.

Inverting Schmitt Trigger Example

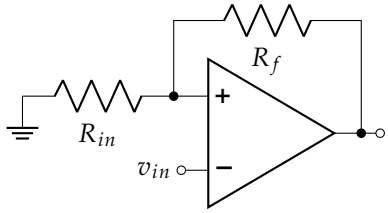

Figure 5.13: Inverting Schmitt Trigger

For the inverting Schmitt Trigger in Figure 5.13, its threshold voltage V_+ is changed by the op amp output as

$$V_+ = v_{out}\frac{R_{in}}{R_{in}+R_f}$$

calculated as a voltage divider with v_{out} alternating between sources $+V_s$ and $-V_s$. Therefore, the threshold triggers at $\pm V_S$ are when

$$v_{in}=V_+=\pm V_s\frac{R_{in}}{R_{in}+R_f}$$

This alternates the threshold voltage. Figure 5.14 shows this behavior for a resistor ratio $R_{in}/(R_{in}+R_f)$ of 0.5 with a supply voltages of ±1 volt, producing upper and lower threshold voltages of ±0.5 volts.

When v_{in} is higher than 0.5, v_{out} hits negative saturation requiring the input to reach the lower threshold before flipping to positive saturation. When v_{in} is lower than 0.5, v_{out} hits positive saturation requiring the input to reach the higher threshold before flipping to negative saturation.

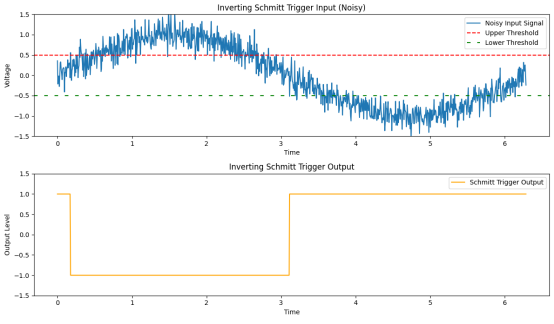

Figure 5.14: Inverting Schmitt Trigger Noisy Sinusoidal Input and Corresponding Output Signals
Code was generated by ChatGPT (OpenAI) in response to a user query in November 2025. The model produced a Python script for simulating the output of an ideal inverting Schmitt Trigger op-amp driven by a noisy sine-wave input.

5.5 Learning Targets

Ultimately operational amplifiers are the building blocks of analog circuits, performing signal manipulation. They amplify voltage with little current and perform mathematical operations on signals. Additionally their high impedance prohibits source altering.

To analyze op amps, first determine the type of feedback. No feedback op amps are comparators, and positive feedback op amps are Schmitt Triggers. Both operate at the saturation limits. Negative feedback op amps operate in the linear region and are analyzed using nodal analysis with ideal op amp assumptions.

With operational amplifiers in the toolbox you can now

- describe what op amps can do.
- discuss differences between ideal and nonideal op amps and when to use each model.
- analyze ideal op amps to find input-output relationships.
- analyze cascading op amp circuits.
- understand and interpret nonideal op amps outputs.

Capacitors and Inductors 6

Capacitors and inductors are fundamental passive circuit elements that store energy rather than dissipate it, making them essential for understanding dynamic circuit behavior. Unlike resistors, which respond instantaneously to applied voltages, these components introduce time-dependent relationships between voltage and current.

While reading the chapter focus on learning how to

- describe what they are made of and what they are designed to do,
- mathematically describe the relationship between and calculate capacitance/inductance, charge, current, voltage, energy, and power,
- implement capacitor and inductor properties and elemental relationships to analyze voltages and currents under dc steady-state conditions,
- find equivalent capacitance and inductance in series and parallel, and
- analyze op amps with capacitors and inductors.

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6.1 Capacitors

A capacitor is a two-terminal passive circuit element designed to store energy. It consists of two conducting plates separated by a dielectric (electrical insulator) like air or ceramic. It stores energy in the electric field between plates. Its circuit symbol, shown in Figure 6.1, resembles plates separated by air. Capacitors may be polarized with defined positive and negative terminals though many are bipolar (nonpolarized) like resistors.

Let's give a capacitor some energy with a voltage source. The source will deposit positive charge q on one plate and negative charge $-q$ on the other, thus storing electric charge. The amount of charge stored is

$$q = Cv \tag{6.1}$$

where q is charge in units of Coulombs, v in volts, and C is the constant of proportionality called **capacitance** with units of Farads (F). So we define C as the ratio of charge on one plate of a capacitor to the voltage difference between the two plates; however, it actually depends on a capacitor's dimensions, specifically plate surface area, permittivity of dielectric material, and distance between plates. Thus, capacitance C is a property.

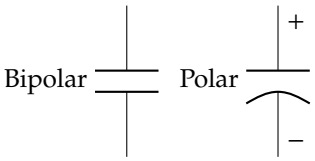

Figure 6.1: Capacitor Circuit Symbols

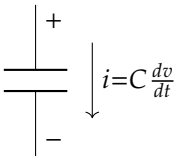

Figure 6.2: Capacitor Passive Sign Convention

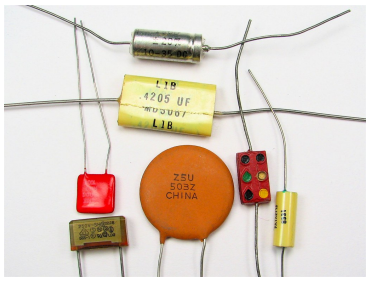

Figure 6.3: Capacitors
Windell Oskay, CC BY 2.0 via Flikr

6.1.1 Elemental Relationship

Given capacitance, the elemental relationship between current and voltage can be derived by differentiating Equation 6.1

$$\frac{dq}{dt} = \boxed{i = C\frac{dv}{dt}} \tag{6.2}$$

Just like a resistor, current direction and polarity matter! Passive sign convection follows that of resistors where Equation 6.2 applies given passive sign convention shown in Figure 6.2.

One can integrate 6.2 and rearrange to get voltage

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt = \frac{1}{C}\int_{t_0}^{t} i(t)dt + v(t_0)$$

where $v(t_0)$ is the voltage across the capacitor at time t_0.

6.1.2 Power and Energy

Ideal capacitors do not dissipate power; energy is captured and delivered not lost. But real capacitors have parasitic effects that dissipate power, primarily through heat or leakage. We will treat them as ideal, which works well for dc steady state and low frequencies (like standard 60 Hz power).

Capacitor instantaneous power $P = iv$ is

$$P = iv\frac{dv}{dt} \tag{6.3}$$

And integrating power gives the energy

$$w = C\int_{-\infty}^{t} v\frac{dv}{dt}dt = C\int_{-\infty}^{t} v\,dv = \frac{1}{2}Cv^2\Big|_{v(-\infty)}^{v(t)}$$

which for an initially uncharged capacitor evaluates to

$$w = \frac{1}{2}Cv^2 = \frac{q^2}{2C} \tag{6.4}$$

6.1.3 Capacitor Properties

An important property of capacitors is that **voltage across a capacitor must be continuous**; it cannot change instantaneously. Instantaneous voltage change means $dv/dt = \infty$, which by its elemental relation demands a physically impossible infinite current. However, current through a capacitor can change instantaneously. Therefore, a capacitor resists abrupt changes in voltage across it.

When a dc source is applied to a capacitor, current flows as charges migrate until the capacitor is fully charged. Once in steady-state (fully charged or discharged), no additional charge transfers in or out of the capacitor, and it blocks current since $dq/dt = 0$. Therefore, a **capacitor is an open circuit to dc steady state** as it cannot carry current under constant steady-state voltage ($dv/dt = 0$). Imagine the capacitor circuit symbol being stretched apart creating an open circuit to remember its steady-state behavior.

6.1.4 Series and Parallel Capacitors

Imagine a current source of current i_{total} in parallel with N parallel connected capacitors. All the capacitor voltages are the same (KVL), and from KCL all their currents summed total the source current so

$$i_{total} = \sum_{k=1}^{N} i_k = \sum_{k=1}^{N} C_k\frac{dv_k}{dt} = \left(\sum_{k=1}^{N} C_k\right)\frac{dv}{dt} = C_{eq}\frac{dv}{dt}$$

Therefore, capacitors connected in parallel can be combined into an equivalent capacitance of

$$C_{eq} = \sum_{k=1}^{N} C_k = C_1 + C_2 + \ldots + C_N \tag{6.5}$$

For capacitors in series, currents are the same so summing voltage gives

$$\frac{d}{dt}\sum_{k=1}^{N} v_k = \left(\frac{dv}{dt}\right)_{total} = \sum_{k=1}^{N}\frac{1}{C_k}i = \frac{1}{C_{eq}}i$$

Therefore, capacitors connected in series combine into an equivalent capacitance of

$$\frac{1}{C_{eq}} = \sum_{k=1}^{N}\frac{1}{C_k} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N} \tag{6.6}$$

This is the opposite of resistors!

6.2 Inductors

An inductor is a two terminal passive circuit element designed to store energy. It consists of an electric conductor, like wire, wound in a coil around a core. It stores energy in a magnetic field. Its circuit symbol, either shown in Figure 6.4 may be used, resembles coiled wire. Like resistors, inductors are not polarized, but they may include a phase dot placed on one of the terminals to indicate the start of the winding.

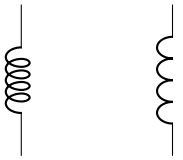

Figure 6.4: Inductor Circuit Symbols

Current generates a magnetic field, and changes in current induce voltage opposing the change. **Inductance** L is defined as the opposition to the change of current flowing through it. It is measured in Henrys (H) and depends on the number of turns, core permeability, cross-sectional area, and length of the inductor. Thus inductance L is a property.

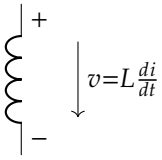

Figure 6.5: Inductor Passive Sign Convention

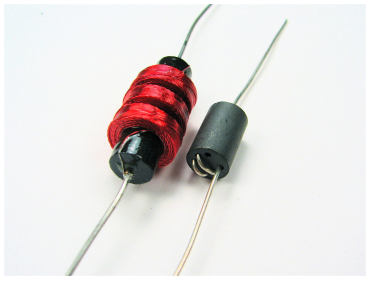

Figure 6.6: Inductors
Phreneticc, CC BY 2.0 via Wikimedia Commons

6.2.1 Elemental Relationship

The voltage across an inductor is directly proportional to the time rate of change of current with inductance as its constant of proportionality

$$\boxed{v = L\frac{di}{dt}} \tag{6.7}$$

Just like a resistor, current direction and polarity matter! Passive sign convection follows that of resistors where Equation 6.7 applies given passive sign convention shown in Figure 6.5.

One can integrate to get current

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)dt = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i(t_0)$$

where $i(t_0)$ is the voltage across the capacitor at time t_0.

6.2.2 Power and Energy

Ideal inductors do not dissipate energy. Real inductors have parasitic effects that dissipate power like winding resistance and capacitance. For now, we will treat them as ideal as parasitic effects are often small.

The instantaneous power $P = iv$ is

$$P = iL\frac{di}{dt} \tag{6.8}$$

And integrating power gives the energy

$$w = L\int_{-\infty}^{t} i\frac{di}{dt}dt = L\int_{-\infty}^{t} i\,di = \frac{1}{2}Li^2\Big|_{i(-\infty)}^{i(t)}$$

which for an initially uncharged inductor evaluates to

$$w = \frac{1}{2}Li^2 \tag{6.9}$$

6.2.3 Inductor Properties

An important property of inductors is that **current through an inductor must be continuous**; it cannot change instantaneously as that would require infinite voltage ($di/dt = \infty$). But the voltage can change instantaneously. Therefore, an inductor resists abrupt changes in current.

When a dc source is applied to an inductor, energy is stored in its magnetic field as current changes. Once fully charged or fully discharged (steady-state), the current is constant as no additional energy flows in or out of the inductor, and its voltage potential drops to zero since $di/dt = 0$. Therefore, an **inductor is a short circuit (wire) to dc steady state** as it cannot sustain a voltage potential. Imagine the inductor circuit symbol being stretched apart creating a wire to remember its steady-state behavior.

Δ-Y and Π-T Transformations

Provided all elements are the same type the same, Δ-Y and Π-T transformations apply to inductors just as they do for resistors. For capacitors, the transforms apply in reverse when replacing the resistors in Figure 4.13 and Figure 4.14.

Y to Δ / T to Π

$$C_a = \frac{C_2C_3}{C_1 + C_2 + C_3}$$
$$C_b = \frac{C_1C_3}{C_1 + C_2 + C_3}$$
$$C_c = \frac{C_1C_2}{C_1 + C_2 + C_3}$$

Δ to Y / Π to T

$$C_1 = \frac{C_aC_b + C_aC_c + C_bC_c}{C_a}$$
$$C_2 = \frac{C_aC_b + C_aC_c + C_bC_c}{C_b}$$
$$C_3 = \frac{C_aC_b + C_aC_c + C_bC_c}{C_c}$$

6.2.4 Series and Parallel Inductors

Imagine a voltage source of voltage v_{total} in series with N series connected inductors. All the inductor currents are the same (KCL), and from KVL all their voltages summed total the source voltage as

$$v_{total} = \sum_{k=1}^{N} v_k = \sum_{k=1}^{N} L_k \frac{di_k}{dt} = \left(\sum_{k=1}^{N} L_k\right) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$

Therefore, inductors connected in series can be combined into an equivalent inductance of

$$L_{eq} = \sum_{k=1}^{N} L_k = L_1 + L_2 + \ldots + L_N \tag{6.10}$$

For inductors in parallel, their voltages are the same so summing currents gives

$$\frac{d}{dt} \sum_{k=1}^{N} i_k = \left(\frac{di}{dt}\right)_{total} = \sum_{k=1}^{N} \frac{1}{L_k} v = \frac{1}{L_{eq}} v$$

Therefore, inductors connected in parallel can be combined into an equivalent inductance of

$$\frac{1}{L_{eq}} = \sum_{k=1}^{N} \frac{1}{L_k} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N} \tag{6.11}$$

This is the same as resistors.

6.3 DC Effective Circuits

For very short time intervals (milliseconds), capacitors and inductors induce transients where voltage and current change with time. After a transient has settled to steady-state, the circuit is considered effectively under dc conditions. In dc effective circuits, capacitors act as open circuits and inductors act as short circuits.

To analyze dc effective circuits, draw capacitors and inductors as how they act but mark the open circuit capacitors with a voltage difference across it and the short circuited inductors with current through it. For example, in Figure 6.7, the voltage of the capacitor under dc steady-state conditions is found by redrawing the circuit as shown in Figure 6.8. Analyzing Figure 6.8, the capacitor voltage is the same as v_2 since they are in parallel. v_2 is found using KCL and KVL and Ohm's Law or using the divider circuit shortcut to be $v_2 = v_c = V_s R_2/(R_1 + R_2)$. The current of the inductor is the same as both resistors and is found using Ohm's Law for R_2 as $i_L = V_s/(R_1 + R_2)$.

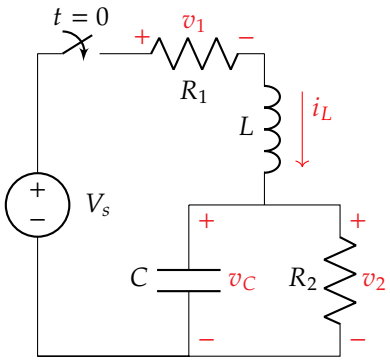

Figure 6.7: Example for DC Effective Circuit

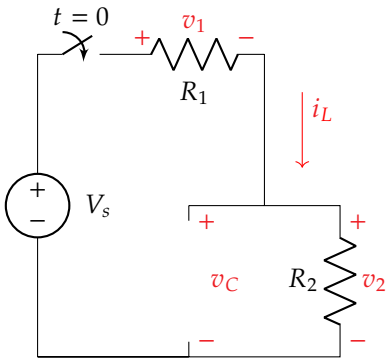

Figure 6.8: Example for DC Effective Circuit Under DC Steady-State Conditions

6.4 Integrator and Differentiator Op Amps

With inductors and capacitors, operational amplifiers can perform calculus on signals with applications from audio filters to PID controllers in control systems. Let's look at two basic op amps that use a capacitor and an inductor as an example of how to analyze op amp circuits with these two new circuit elements.

6.4.1 Integrator Op Amp

For an inverting op amp with a capacitor in the feedback path, the output of the op amp in Figure 6.9 is proportional to the integral of the input signal. This is found using nodal analysis at the v_- node with Equation 6.2 for the feedback current path.

$$\frac{v_- - V_s}{R} + i_C = \frac{v_- - V_s}{R} + C\frac{d(v_- - v_{out})}{dt} = 0 \tag{6.12}$$

Recall for an ideal op amp $v_- = v_+$, which in this case is zero, yielding

$$\frac{0 - V_s}{R} + C\frac{d(0 - v_{out})}{dt} = 0$$

$$\frac{dv_{out}}{dt} = -\frac{1}{RC}V_s$$

Integrating to find the output voltage assuming $v_{out}(t = 0) = 0$ gives

$$v_{out} = -\frac{1}{RC}\int_0^t V_s dt \tag{6.13}$$

Thus, the output is the integral of the input. If the input is a square wave, an integrator op amp outputs a triangle wave with slopes equal to the amplitude of the square wave.

The topic of operational amplifiers may be covered out of order anytime after finishing DC Circuit Analysis Tools, especially Chapter 2. If op amps have not already been covered, go back and read Chapter 5.

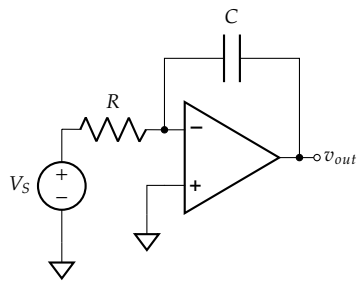

Figure 6.9: Basic Integrator Op Amp

6.4.2 Differentiator Op Amp

Swapping the resistor and capacitor positions of the integrator op amp in Figure 6.9 builds a differentiator op amp. Replacing the capacitor with an inductor also builds a differentiator op amp. The output of the op amp in Figure 6.10 is proportional to the rate of change (derivative) of the input signal. The output is found using nodal analysis at the v_- node with Equation 6.7 integrated for the feedback current path.

$$\frac{v_- - V_s}{R} + i_L = \frac{v_- - V_s}{R} + \frac{1}{L}\int_0^t (-v_- - V_{out})dt = 0 \tag{6.14}$$

Recall for an ideal op amp $v_- = v_+$, which in this case is zero, yielding

$$\frac{-V_s}{R} + \frac{1}{L}\int_0^t -v_{out}dt = 0$$

$$\int_0^t v_{out} = -\frac{L}{R}V_s$$

Differentiating to find the output voltage gives

$$v_{out} = -\frac{L}{R}\frac{dV_s}{dt} \tag{6.15}$$

Thus the output is the derivative of the input. If the input is a triangle wave, The a differentiator op amp output will be a square wave with its amplitudes equal ot the slopes of the triangle wave.

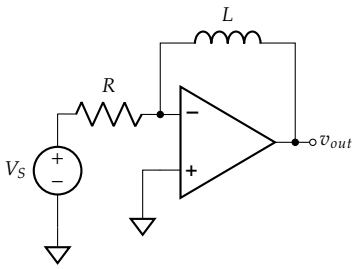

Figure 6.10: Basic Differentiator Op Amp

6.5 Learning Targets

After learning the fundamentals of capacitors and inductors, you can

- describe what they are made of and what they are designed to do,
- mathematically describe the relationship between and calculate capacitance/inductance, charge, current, voltage, energy, and power,
- implement capacitor and inductor properties and elemental relationships to analyze voltages and currents under dc steady-state conditions,
- find equivalent capacitance and inductance in series and parallel, and
- analyze op amps with capacitors and inductors.

Since capacitor and inductor current-voltage relationships depend on time unless under dc steady state, analysis as a function of time is next.

First Order Circuits 7

With the introduction of capacitors and inductors, voltages and currents are no longer constant but change over time. After a perturbation like switching on a power source, voltages and currents will change. This short lived change before reaching a different steady-state equilibrium is called a transient. It occurs because capacitors and inductors cannot change their energy state instantly. The circuit equations modeling transients have derivatives (or integrals)—that is, they are differential equations.

While reading the chapter focus on learning how to

- formulate circuit differential equations using KVL and KCL,
- list and apply the steps to solve (1st order) circuit differential equations
- determine currents and voltages in 1st order circuits (transient circuit response), and
- draw/sketch graphical solutions to transients.

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7.1 Circuit Differential Equations

The general strategy for formulating circuit differential equations remains the same as resistive circuits. Namely, apply KCL and KVL (the basis of all our circuit analysis tools) and use each element's current-voltage relationship.

For resistors, the elemental response is Ohm's Law

$$v_R = i_R R \tag{7.1}$$

which will now be denoted with subscripts to indicate its corresponding circuit element. For capacitors, it is

$$i_c = C\frac{dv_c}{dt} \tag{7.2}$$

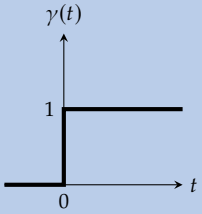
and for inductors it is

$$v_L = L\frac{di_L}{dt} \tag{7.3}$$

which include derivatives.

Step Response

The step response function is often denoted by $u(t)$ but γ is used instead so that it's LaPlace transform variable Γ looks like the step function. Before time $t = 0$, it is zero and at time $t = 0$ it jumps up to one.



With these differentials, a circuit's response is transient, meaning it has time varying behavior that occurs after a system is perturbed. The transient eventually settles to equilibrium or steady-state.

In circuits, transients are most often caused by switches, which create a step response due to step function $\gamma(t)$ introduced by the switch.

Steps to Solve Circuit Differentials

There is a six step process for solving circuit differentials in order to find the time varying response of a desired element. The first step is to formulate a circuit's differential equation (diff EQ) for that desired quantity, typically the voltage or current of an inductor or capacitor. All our circuit differential equations have constant coefficients and are linear in the independent variable. The solution of these are well established in mathematics and constitute most of the remaining steps to solve for the unknown variable as a function of time. The final step requires using circuit analysis tools to find initial conditions (ICs) to be applied.

With voltage or current of one element in a circuit over time known, all other current or voltage quantities of interest can be determined using elemental relationships in Equations 7.1, 7.2, and 7.3 along with circuit analysis equations (KCL&KVL) used in the first step. Therefore, it is unnecessary to find a differential equation specifically for the quantity of interest. The differential equation may be for any unknown, and then with that solution, the desired unknown can be found.

7.1.1 Step 1: Formulate Differential Equation

Formulate a differential equation for a single unknown by analyzing the circuit immediately after the perturbation occurs using Kirchhoff's Current Law and Kirchhoff's Voltage Law (KCL&KVL) and elemental relationships in Equations 7.1, 7.2, and 7.3. Let's look at some examples.

Example 7.1.1 RC Series Circuit

For the RC series circuit shown in Figure 7.1, first draw the current direction and polarity for each element. Apply KCL&KVL to the circuit when the switch has just closed to yield

$$\begin{aligned} \text{KCL:} \quad & i = i_R = i_c \\ \text{KVL:} \quad & -V_s + v_R + v_C = 0 \end{aligned}$$

where current is the same through all elements in series so its subscripts can be dropped.

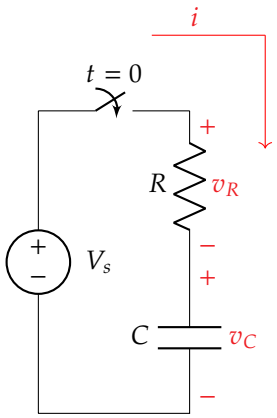


Figure 7.1: RC Series Circuit

Starting with KVL, let's reduce equation to a single unknown using elemental relations.

$$v_R = i_R R \;\rightarrow\; V_s = iR + v_C$$

where the subscript for current can be dropped.

Then we can use the capacitor elemental relation for current to yield

$$i_c = C\frac{dv_c}{dt} \;\rightarrow\; V_s = RC\frac{dv_C}{dt} + v_C$$

which is a differential equation with a single unknown v_c. After solving this equation for v_c, one could find v_R using KVL and current by differentiating or using Ohm's law with v_R now known. Alternatively, we could have differentiated KVL on both sides with respect to time so the $\frac{dv_c}{dt}$ could be replaced with $\frac{i}{C}$. yielding

$$\begin{aligned} \frac{dV_s}{dt} = 0 &= R\frac{di}{dt} + \frac{dv_c}{dt} \\ 0 &= R\frac{di}{dt} + \frac{i}{C} \\ 0 &= RC\frac{di}{dt} + i \end{aligned}$$

which is a first order differential equation with a single unknown i, and since V_s is a constant its derivative is zero. After solving this equation for i, one could find v_R using Ohm's Law and then v_C using KVL.

Each differential equation is first order because the order of the highest derivative is one. Note that RC has dimensions of time. We will see it is the circuit's time constant.

Known vs Unknown Varible Confusion

If it's hard to know when you have a single unknown variable because all the known circuit quantities are represented as variables try using numbers as variables! Take each known and assign it a numerical value but do not use any numerical values as numbers. Do not do math with the numerical variables! Prime numbers are recommended to reduce the inclination to mathematically simplify numerical variables. For Example 7.1.1, using $V_s = 3$, $R = 5$, and $C = 7$, it is reworked as

$$\begin{aligned} V_s = &\quad iR + v_C \\ 3 = &\quad 5i + v_C \\ i_c = 7\frac{dv_c}{dt} \;\rightarrow 3 = &\quad 5 \cdot 7\frac{dv_C}{dt} + v_C \end{aligned}$$

clearly showing when a single unknown variable remains.

Example 7.1.2 RL Parallel Circuit

For the RL parallel circuit shown in Figure 7.2, first draw the current direction and polarity for each element. Apply KCL&KVL to the circuit when the switch has just closed to yield

$$\begin{aligned} \text{KCL:} \quad & I_s = i_R + i_L \\ \text{KVL:} \quad & v = v_R = v_L \end{aligned}$$

where voltage is the same across all elements in parallel so its subscripts can be dropped.

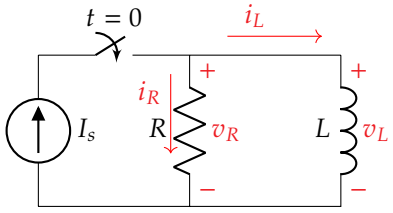


Figure 7.2: RL Parallel Circuit

Starting with KCL let's reduce the equation to a single unknown using elemental relations.

$$v_R = i_R R \;\rightarrow\; I_s = \frac{v}{R} + i_L$$

where the subscript for voltage can be dropped.

Then we can use the elemental relation for inductors to yield

$$v_L = L\frac{di_L}{dt} \;\rightarrow\; I_s = \frac{L}{R}\frac{di_L}{dt} + i_L$$

which is a first order differential equation with a single unknown i_L. After solving this equation for i_L, one could find i_R using KCL and voltage by differentiating or using Ohm's law with i_R now known. Alternatively, we could have differentiated both sides with respect to time so the $\frac{di_L}{dt}$ could be replaced with $\frac{v_L}{L}$ yielding

$$\begin{aligned}\frac{dI_s}{dt} = 0 &= \frac{1}{R}\frac{dv}{dt} + \frac{di_L}{dt}\\ 0 &= \frac{1}{R}\frac{dv}{dt} + \frac{v_L}{L}\\ 0 &= \frac{L}{R}\frac{dv}{dt} + v\end{aligned}$$

which is a first order differential equation with a single unknown v, and since I_s is a constant its derivative is zero. After solving this equation for v, one could find i_R using Ohm's Law and then i_L using KCL.

Note that R/L has dimensions of time. We will see it is the circuit's time constant.

Example 7.1.3 RLC Series Circuit

For the RLC series circuit shown in Figure 7.3, first draw the current direction and polarity for each element. Apply KCL&KVL to the circuit when the switch has just closed to yield

$$\begin{aligned}&\text{KCL:} && i = i_R = i_c = i_L\\ &\text{KVL:} && -V_s + v_R + v_C + v_L = 0\end{aligned}$$

where current is the same through all elements in series so its subscripts can be dropped. Starting with KVL let's reduce equation to a single unknown using elemental relations.

$$\begin{aligned}v_R &= i_R R\\ v_L &= L\tfrac{di_L}{dt}\end{aligned} \;\rightarrow\; V_s = iR + v_C + L\frac{di}{dt}$$

where the subscripts for current can be dropped. Thus, we can use another elemental relation *and its derivative* to yield

$$\begin{aligned}i_c &= C\tfrac{dv_c}{dt}\\ \tfrac{di_c}{dt} &= C\tfrac{d^2v_c}{dt^2}\end{aligned} \;\rightarrow\; V_s = RC\frac{dv_C}{dt} + v_C + LC\frac{d^2v_C}{dt^2}$$

which is a differential equation with a single unknown v_c. After solving this equation for v_c as shown in Section 8.2, one could find i_C by differ-

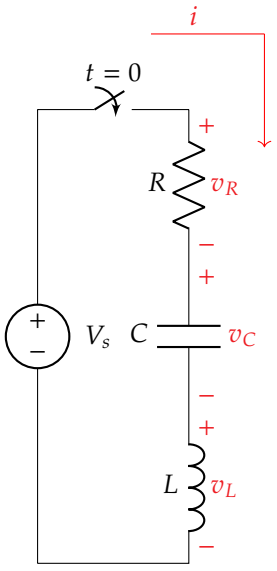


Figure 7.3: RLC Series Circuit

entiating, v_R using Ohm's law, and lastly v_L using KVL. Alternatively, we could have differentiated both sides with respect to time so the $\frac{dv_c}{dt}$ could be replaced with $\frac{i}{C}$ yielding

$$\begin{aligned}\frac{dV_s}{dt} = 0 &= R\frac{di}{dt} + \frac{dv_c}{dt} + L\frac{d^2i}{dt^2} \\ 0 &= R\frac{di}{dt} + \frac{i}{C} + L\frac{d^2i}{dt^2} \\ 0 &= RC\frac{di}{dt} + i + LC\frac{d^2i}{dt^2}\end{aligned}$$

which is a differential equation with a single unknown i, and since V_s is a constant its derivative is zero. After solving this equation for i, one could find v_L by differentiating, v_R using Ohm's Law, and then v_C using KVL.

Each differential equation is second order because the order of the highest derivative is two. When both inductors and capacitors are present (two energy sources), the differential equation is now second order whereas when only one is present it is first order. Also note that LC has dimensions of time squared.

Math vs Engineering Variables

Typically mathematicians use different variables than engineers. For example, in math you solve for y with respect to x, and in electronics we will solve for x or y with respect to time t where x is the unknown voltage or current of an element. Other differences include using s instead of r for roots of the characteristic equation since R is resistance, a or A instead of c or C for constants since C is for capacitance, and j instead of i for the imaginary number since i is current. See summary in Table 7.1.

Table 7.1: Math vs Engineering Variables

	Mathematics	Engineering
Dependent Variable	y	x=i, v
Independent Variable	x	t
Variable Derivatives	$\frac{dy}{dx}$	$\frac{dx}{dt}$, $\frac{dy}{dt}$
Characteristic equation roots	r	s
Constants	C	a, A, B
Imaginary number	i	j

7.1.2 Step 2: Write Differential Equation in General Form

The standard or general form for a linear, constant coefficient differential equation is

$$a_0x + a_1\frac{dx}{dt} + a_2\frac{d^2x}{dt^2} + \ldots = B \tag{7.4}$$

where a_0, a_1, a_2 etc. and B are constants, x is the unknown dependent variable representing a physical voltage or current in the circuit, and t is the independent variable time. These variables are defined by matching the form to the formulated differential equation in Step 1.

7.1.3 Step 3: Transform x to $y = x - B/a_0$

If $B = 0$, the differential equation is homogeneous. However, if $B \neq 0$, the differential equation is inhomogeneous and requires a particular solution of its inhomogeneous equation in addition to the general or complementary solution of the homogeneous equation. However, since B is a constant, we can instead perform a transformation to make the equation homogeneous.

To transform, let $y = x - \frac{B}{a_0}$ and thus

$$\begin{aligned}x &= y + \frac{B}{a_0} \\ \frac{dx}{dt} &= \frac{dy}{dt}\end{aligned}$$

Dividing Equation 7.4 by a_0 and substituting yields

$$\frac{a_0 x}{a_0} + \frac{a_1}{a_0}\frac{dx}{dt} + \frac{a_2}{a_0}\frac{d^2x}{dt^2} + \ldots = \frac{B}{a_0}$$
$$y + \frac{\cancel{B}}{\cancel{a_0}} + \frac{a_1}{a_0}\frac{dy}{dt} + \frac{a_2}{a_0}\frac{d^2y}{dt^2} + \ldots = \frac{\cancel{B}}{\cancel{a_0}}$$
$$y + \frac{a_1}{a_0}\frac{dy}{dt} + \frac{a_2}{a_0}\frac{d^2y}{dt^2} + \ldots = 0$$

which is now a homogeneous equation. Whereas x referred to a voltage or current in the circuit, this new unknown y has no physical meaning!

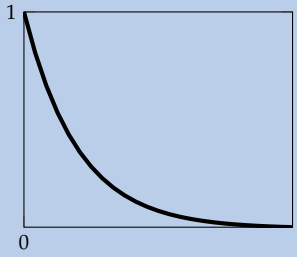
If $B = 0$ the transformation becomes $y = x$, therefore this step can be applied even if $B = 0$ or it can be skipped.

Time Shift Transformation

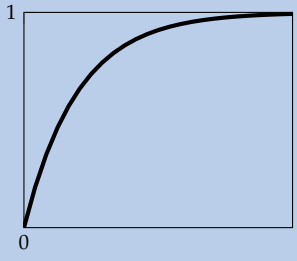
Note that the perturbation can be shifted from $t = 0$ to $t = t_0$ via a t to $t - t_0$ transform.
Define a new variable $T = t - t_0$, which means $t = T + t_0$, and a new dependent variable $h(T) = y(t) = y(T + t_0)$.
The transform of the derivatives remains the same so $dy/dt = dh/dT$ and $dy^2/dt^2 = dh^2/dT^2$ since $dT = d(t + t_0) = dt$.
The initial conditions must be transformed too! So $y(t_0) = y_0$ becomes $h(T = 0) = y_0$.

Decaying Exponential

The shape of a decaying exponential $y = e^{-t/\tau}$ starts at the value of one and decays to zero. The speed of the decay depends on the time constant τ, which is defined as the time to decay to 37% $(1/e)$ of its initial value. One sees this shape for discharging a capacitor.



The response may instead be due to a step input from a source like for charging of a capacitor with a voltage source. In this case, the decaying exponential shape is present but is subtracted from the step input as $y = 1 - e^{-t/\tau}$. The time constant τ is defined as the time to reach 63% $(1 - 1/e)$ of its final value.



7.1.4 Step 4: Find General Solution to General Form

The general solution to the homogeneous form

$$a_0 y + a_1 \frac{dy}{dt} + a_2 \frac{d^2y}{dt^2} + \ldots = 0$$

is well known to be the sum of exponentials. For a first order system, only the terms up to the first derivative remain

$$a_0 y + a_1 \frac{dy}{dt} = 0$$

and its solution is $y = Ae^{st}$ where to be a solution must satisfy the characteristic equation

$$a_0 s^0 + a_1 s^1 + a_2 s^2 + \ldots = 0 \tag{7.5}$$

The characteristic equation sums each coefficient multiplied by s raised to the power of the coefficient term's derivative order. For first order, the characteristic equation is

$$a_0 + a_1 s = 0$$

Solving for the root s yields

$$s = -\frac{a_0}{a_1} = -\frac{1}{\tau}$$

where τ is the time constant. The solution becomes

$$y = Ae^{-t/\tau} = Ae^{-\frac{a_0}{a_1}t}$$

where A is an unknown constant.

Passive systems always have solutions with decaying exponentials (negative exponent) as the circuit damps or decays back to zero or its steady-state dc level that was transformed out in Step 3.

7.1.5 Step 5: Transform Back to $x = y + B/a_0$

With the homogeneous solution now known, we want to transform back from our placeholder variable y to x, which has physical meaning. Do not be tempted to apply the initial conditions (ICs) before this step as y is a placeholder variable. It has no physical meaning so ICs cannot be determined for y. Transforming back, the solution form becomes

$$x = Ae^{-\frac{a_0}{a_1}t} + \frac{B}{a_0} = Ae^{-t/\tau} + \frac{B}{a_0} \tag{7.6}$$

7.1.6 Step 6: Apply Initial Conditions

The last step is to apply initial conditions (ICs) to solve for the unknown constants. For first order differential equations, there is only one unknown constant A and thus only one IC $x(t = 0^+)$ is needed at $t = 0^+$ immediately after the perturbation. For higher order equations, an IC is needed for each order (e.g. three ICs for third order).

Steps 2-5 were just math. This step is just math once we know the initial conditions. Namely, find the unknown constant A using $x(t = 0^+) = X_0$ substituting $t = 0$ and the known value of X_0 into Equation 7.6

$$x(t = 0^+) = X_0 = Ae^0 + \frac{B}{a_0} \rightarrow A = X_0 - \frac{B}{a_0}$$

which yields a final solution of

$$x(t) = \left(X_0 - \frac{B}{a_0}\right)e^{-\frac{a_0}{a_1}t} + \frac{B}{a_0} = \left(X_0 - \frac{B}{a_0}\right)e^{-t/\tau} + \frac{B}{a_0} \tag{7.7}$$

How to Find ICs

If the ICs are unknown, circuit analysis and applying the properties of inductors and capacitors will determine them. Recall that capacitors cannot have abrupt changes in voltage and inductors cannot have abrupt changes in current. Therefore, the voltage of a capacitor and current of an inductor immediately before a perturbation of the system at $t = 0^-$ must be equal immediately after the perturbation at $t = 0^+$ which is written mathematically as

$$v_C(0^-) = v_C(0^+) \tag{7.8}$$
$$i_L(0^-) = i_L(0^+) \tag{7.9}$$

Remark 7.1.1 Capacitor current and inductor voltage can and often do change abruptly. Therefore, do not be tempted to equate their values before and after a perturbation.

$$i_C(0^-) \neq i_C(0^+)$$
$$v_L(0^-) \neq v_L(0^+)$$

Before the system's perturbation, the circuit is under dc steady-state conditions and capacitors are treated as open circuits and inductors are treated as short circuits.

Final Values

Similar to finding initial conditions/values, final values may be found by drawing the circuit after the perturbation but assuming the transient has finished and the circuit is under dc steady-state conditions.

Therefore, to find initial conditions

1. Draw the circuit under dc steady-state conditions before the system's perturbation and find the voltages of the capacitors (across its open circuit location) and currents of the inductors (through its short circuit location).

2. Use Equations 7.8 and 7.9, with KCL&KVL of circuit right after perturbation ($t = 0^+$) if needed, to find the desired initial condition $x(t = 0^+)$ right after the perturbation.

Sketching Transient Response

For all transient response sketches, the initial value is constant (flat line) from time zero and before. At time zero, the transient starts and follows either a decaying exponential e^{-t} or $1 - e^{-t}$ shape settling to a constant final or steady-state value (approaching a flat line).

7.2 Charging a Capacitor Example

In Example 7.1.1, the RC series circuit in Figure 7.1 charges a capacitor when the switch gets closed at time $t = 0$. Assume that before the switch closes the capacitor voltage is known to be V_0. Let's find the voltage of the capacitor over time due to the perturbation of the switch closing by following the steps to solve circuit differential equations.

1. Review how the differential equation was formulated in Section 7.1.1 using KCL&KVL.

2. It is already in standard form

$$v_c + RC\frac{dv_c}{dt} = V_s$$

3. Transform the inhomogeneous equation for the unknown $x(t) = v_C(t)$ to a homogeneous equation.

$$\begin{aligned} y(t) &= x(t) - B/a_0 \\ y &= x - V_s \\ \frac{dy}{dt} &= \frac{dx}{dt} \end{aligned}$$

where x is v_C and plugged into the differential equation yields

$$\begin{aligned} y + V_s + RC\frac{dy}{dt} &= V_s \\ y + RC\frac{dy}{dt} &= 0 \end{aligned}$$

which is now homogeneous.

4. The solution to a homogeneous linear constant coefficient equation is $y = Ae^{st}$ which satisfies the characteristic equation $a_0 + a_1 s = 0$. Therefore, the solution form is

$$\begin{aligned} 1 + RCs = 0 \rightarrow s &= -\frac{1}{RC} = -\frac{1}{\tau} \\ y &= Ae^{-\frac{1}{RC}t} = Ae^{-\frac{t}{\tau}} \end{aligned}$$

where the time constant τ is found to be RC.

5. Before applying the IC to find the unknown constant A, transform back.

$$x = y + B/a_0$$
$$x = y + V_s$$
$$x(t) = Ae^{-\frac{1}{RC}t} + V_s$$

Do not be tempted to apply the ICs until transformed back to x as y is a placeholder variable and has no direct physical meaning.

6. Using the IC given as $v_C(t = 0^-) = v_C(t = 0^+) = V_0$, the unknown constant is found as

$$V_0 = Ae^0 + V_s \longrightarrow A = V_0 - V_s$$

and the final solution is

$$v_C(t) = V_s + (V_0 - V_s)e^{-\frac{1}{RC}t}$$

whose shape is sketched in Figure 7.4.

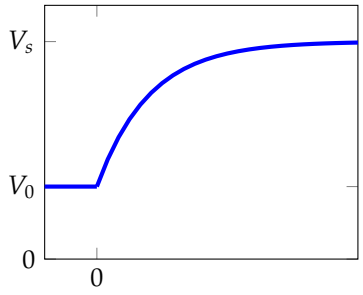


Figure 7.4: Charging Capacitor Voltage Response

If the initial condition was not given then intuitively one assumes the capacitor is fully discharged with zero volts. To verify this, draw the circuit at time $t = 0^-$ under dc steady-state conditions, assuming the switch has been open long enough for previous transients to have decayed to steady-state (Figure 7.5). The capacitors is treated as an open circuit. Using KCL, the current is zero because the switch is open. Without a closed path, KVL does not apply so v_R and $v_C(0^-)$ are zero. The IC is $v_C(0^+) = v_C(0^-) = 0$, which shifts the response in Figure 7.4 down to zero at time $t = 0$ and before.

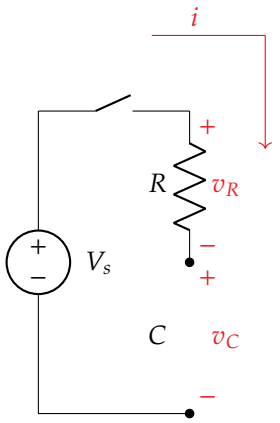


Figure 7.5: RC Series Circuit Under DC Steady-State Conditions

Let's find other voltages and currents. With $v_C(t)$, KVL gives

$$v_R(t) = V_s - \left(V_s + (V_0 - V_s)e^{-\frac{1}{RC}t}\right) = (V_s - V_0)e^{-\frac{1}{RC}t}$$

Current can then be found using Ohm's law or from applying the steps to solve the differential equation found in Section 7.1.1 with current as the unknown to find

$$i = i_C = i_R = \frac{v_R}{R} = \frac{V_s - V_0}{R}e^{-\frac{1}{RC}t}$$

whose shaped is sketched in Figure 7.6. If $V_0 = 0$, Figure 7.6 shifts up to start at V_s/R but still ends at zero. Notice that capacitor current changes instantaneously, jumping when the perturbation starts at time $t = 0$. Capacitor current does not need to be continuous but its voltage does.

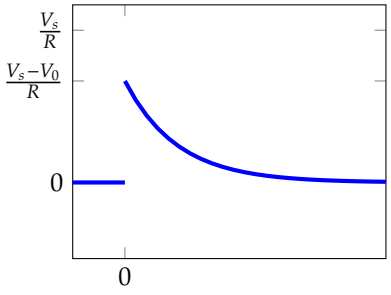


Figure 7.6: Charging Capacitor Current Response

7.3 First Order Shortcut and Solving First Order Recap

First Order Shortcut

Comparing the charging capacitor solution to the general solution final form of Equation 7.7

$$\begin{aligned} v_c(t) &= V_s + (V_0 - V_s)e^{-t/RC} \\ x(t) &= \frac{B}{a_0} + \left(X_0 - \frac{B}{a_0}\right) e^{-t/\tau} \end{aligned}$$

shows a consistent pattern in first order circuit solutions. They are the sum of a transient term proportional to e^{st} (the homogeneous or natural solution) and a final steady-state constant X_{ss} (the forced solution).

Our mathematical solution used the final dc steady-state value in the transformation steps before solving for the transient, and this final dc steady-state value often can be found more quickly to bypass formally solving the differential equation. This suggests a shortcut to finding both coefficients in the solution form $x = A_1 + A_2 e^{st}$ using the IC value X_0 and final steady-state value X_{ss} (dc steady-state value as time approaches infinity).

Therefore, the shortcut for first order circuit solutions is

$$x(t) = X_{ss} + x_{trans} \tag{7.10}$$

$$x(t) = X_{ss} + (X_0 - X_{ss})\, e^{-t/\tau} \tag{7.11}$$

But what is the time constant τ? For RC circuits, $\tau = R_{th}C_{eq}$ and for RL circuits, $\tau = L_{eq}/R_{th}$ where when the capacitors or inductors are combined, the Thevenin resistance is the equivalent resistance seen from its terminals. It can also be found from the circuit differential or characteristic equation to be a_1/a_0.

To take the shortcut,

1. Write down the solution form $x(t) = X_{ss} + (X_0 - X_{ss})\, e^{-t/\tau}$
2. Solve for initial (at $t = 0^+$) and final values (at steady-state as $t \to \infty$).
3. Find the time constant $\tau = R_{th}C_{eq}$ or $\tau = L_{eq}/R_{th}$.

First Order Steps Recap

In short summary, the steps to solve circuit differential equations to find the transient response are

1. Formulate a differential equation using Kirchhoff's Laws and circuit element i-v relationships.
2. Write differential equation in general form.
3. Transform to $y = x - B/a_0$ if $B \neq 0$ and/or via $t = t - t_0$ if perturbation is not at $t = 0$.
4. Use general solution form $y = Ae^{st}$ satisfying characteristic equation $a_0 + a_1 s = 0$
5. Transform back to $x = y + B/a_0$
6. Determine and apply initial condition.

To find initial conditions, draw circuit under dc conditions before perturbation and then remember for capacitors voltage is continuous $v_C(0^-) = v_C(0^+)$ and for inductors current is continuous $i_L(0^-) = i_L(0^+)$, which can be used to find $t = 0^+$ values with KCL&KVL and elemental relations for right after perturbation.

7.4 Learning Targets

By combining Kirchhoff's Laws with differential equation solving techniques from mathematics, you can

- formulate circuit differential equations using KVL and KCL,
- list and apply the steps to solve (1st order) circuit differential equations
- determine currents and voltages in 1st order circuits (transient circuit response), and
- draw/sketch graphical solutions to transients.

7.5 Nonelectrical First Order Systems

The transient behavior of first order circuits is generalizable to other dynamic systems that are first order. Some examples include thermal systems and liquid-level systems.

Thermal System

In a thermal system, the thermal resistance R is the change in temperature difference given the change in heat flow rate or heat transfer rate $\dot{Q}$ through a medium

$$\dot{Q} = \frac{\theta}{R} \tag{7.12}$$

where θ is the temperature difference. This is analogous to Ohm's law written as $i = v/R$ recalling that current i is a flow rate of charge q, and v is a potential difference. So the driving force for heat transfer is the temperature difference and the rate of heat transfer is limited by a medium's thermal resistance just like voltage difference is the driving force for moving charge and the current (rate of charge) is limited by a circuit's resistance.

Heat transfers by conduction, convection, and radiation (though radiation is typically negligible except at very high temperatures and often is encompassed in convective heat transfer coefficients). For conduction, thermal resistance is

$$R_{cond} = \frac{\Delta x}{kA} \tag{7.13}$$

where k is the property thermal conductivity of a medium, A is area normal to heat flow, and Δx is the thickness of the medium. For convection, thermal resistance is

$$R_{conv} = \frac{1}{hA} \tag{7.14}$$

where h is the convective heat transfer coefficient and A is the area normal to heat flow.

Thermal capacitance C is the change in heat stored per change in temperature or

$$C = mc \tag{7.15}$$

where m is mass and c is specific heat of a medium.

The first law of thermodynamics is a heat energy balance applied to a system, which yields a first order differential equation for temperature over time. Consider a conducting medium that can be analyzed as a lumped system where temperature changes over time and not position. The first law balancing heat transfer into medium with change in heat stored in medium yields

$$\dot{Q}_{in} = -\frac{\theta}{R} = mc_p \frac{d\theta}{dt} \;\rightarrow\; C\frac{d\theta}{dt} + \frac{\theta}{R} = RC\frac{d\theta}{dt} + \theta = 0 \tag{7.16}$$

where the heat transfer rate in $\dot{Q}_{in}$ is due to convection ($R = 1/hA$) and θ is the difference in temperature between the surface (and entire conducting medium) and the surrounding convective medium (constant). This differential equation matches the first order homogeneous charging capacitor equation, and thus each system's exhibits the same transient response.

Liquid-Level System

A liquid-level system with flow between two tanks has a resistance of liquid flow R equal to the change in level (height) difference H (a potential) between the two tanks per change in flow rate Q. For smooth laminar flow it is modeled as

$$Q = \frac{H}{R} \tag{7.17}$$

which is analogous to Ohm's Law written $i = v/R$.

The capacitance C of a tank is equal to the change in liquid stored given the change in head (liquid-level height), which is the cross-sectional area A of a tank. Using the principle of continuity (conservation of mass), the flow rate balance yields a first order differential equation modeling liquid-level behavior for a laminar system as the change of fluid volume in the tank

$$\frac{dV}{dt} = A\frac{dh}{dt} = q_i - q_o$$

where V is volume, A is area, h is small deviation in height (level, head), q_i is small deviation of flow in, and q_o is small deviation of flow out. A turbulent system is not linear but can be linearized to model small changes.

Since $A = C$ and $h/q_o = R$ (Equation 7.17), this flow rate balance becomes

$$C\frac{dh}{dt} = q_i - h/R \rightarrow RC\frac{dh}{dt} + h = Rq_i$$

This differential equation is the same as the inhomogeneous charging capacitor equation. Here Rq_i acts as the driving force of changing liquid level analogous to how a voltage source is the driving force for charging a capacitor. Consequently both systems exhibit the same transient response.

Second Order Circuits 8

The steps to solve circuit differentials for second order circuits are the same as for first order circuits in Chapter 7; however, the general form in Step 2 differs slightly. It requires the second order derivative term coefficient be one (a normalized differential equation). This yields a quadratic characteristic equation with two roots. The pair of roots are classified into four types of roots (four solution cases)

1. two complex roots (undamp),
2. two complex-conjugate roots (underdamp),
3. two repeated real roots (critically damp), and
4. two distinct real roots (overdamp),

which characterize the behavior of the transient response, specifically its damping.

While reading the chapter focus on learning how to

- formulate circuit differential equations using KVL and KCL,
- list and apply the steps to solve (2nd order) circuit differential equations
- determine currents and voltages in 2nd order circuits (transient circuit response) for all four types, and
- draw/sketch graphical solutions to transients.

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8.1 Second Order Circuit Solution Steps

The general form for second order circuits has a second order derivative term. Thus, its characteristic equation is quadratic with two roots, and its solution requires two ICs to find its two unknown constants. Compared to first order, Steps 2, 4, and 6 are slightly different but in general the same as first order circuits. Let's see all the steps for second order circuits.

8.1.1 Step 1: Formulate Differential Equation

The first step remains identical regardless of system order. Example 7.1.3 in Subsection 7.1.1 shows an example of how to formulate a differential equation for a single unknown that yields a second order differential equation. It was found by analyzing the circuit right after the perturbation occurs using Kirchhoff's Laws and element i-v relationships.

8.1.2 Step 2: Write Differential Equation in General Form

The standard or general form for a linear differential equation with constant coefficients is

$$a_0 x + a_1 \frac{dx}{dt} + a_2 \frac{d^2x}{dt^2} + \ldots = B \tag{8.1}$$

Since the characteristic equation is quadratic for second order systems, a more practical form normalizes the differential equation to

$$\frac{d^2x}{dt^2} + 2\sigma \frac{dx}{dt} + \omega_n^2 x = B \tag{8.2}$$

where Subsection 8.1.4 (Step 4) illustrates why these defined coefficients are convenient when solving the quadratic characteristic equation.

Additionally, this form introduces two constants with physical significance for the system behavior. The attenuation or exponential decay constant σ determines the rate the response amplitude decreases. The undamped natural frequency ω_n is the system's oscillation rate with no energy loss (damping). They are related as

$$\sigma = \zeta \omega_n \tag{8.3}$$

where ζ is the damping ratio, a dimensionless measure of how oscillations decay after a disturbance. These parameters identify the system response (solution) type in Step 4.

8.1.3 Step 3: Transform x to $y = x - B/a_0$

This step remains the same despite the system order. If $B = 0$, the differential equation is homogeneous. However, if $B \neq 0$, the differential equation is inhomogeneous and requires a particular solution of its inhomogeneous equation in addition to the general or complementary solution of the homogeneous equation. However, since B is a constant, we can instead perform a transformation to make the differential equation

homogeneous. Review Subsection 7.1.3 for derivation of transform. A t to $t - t_0$ transform. for shifting the perturbation from $t = 0$ to $t = t_0$ may be applied just like first order circuits.

8.1.4 Step 4: Find General Solution to General Form

The general solution to the homogeneous form

$$\frac{d^2y}{dt^2} + 2\sigma\frac{dy}{dt} + \omega_n^2 y = 0$$

which is well known to be the sum of exponentials, is

$$y = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{8.4}$$

where A_1 and A_2 are unknown constants and s_1 and s_2 satisfy the characteristic equation

$$s^2 + 2\sigma s + \omega_n^2 = 0 \tag{8.5}$$

where each coefficient is multiplied by s raised to the power of the coefficient term's order. It is solved using the quadratic formula as

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2\sigma \pm \sqrt{(2\sigma)^2 - 4\omega_n^2}}{2} = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2} \tag{8.6}$$

where $a = 1$, $b = 2\sigma$, and $c = \omega_n^2$ conveniently simplifies each term.

Given that the roots include a difference under the square root (discriminant), the solution falls into one of four different types—three common and one special case—based on the sign and value of this difference.

Imaginary Number

In math, i is used for representing the imaginary number, but in electronics i is current. Therefore, use j as the imaginary number. Recall the definition:

$$j := \sqrt{-1}$$

Undamped $\sigma = 0$

Let's examine the special case where σ is zero ($\zeta = \sigma/\omega_n = 0$). The roots of the characteristic equation are purely imaginary

$$s = \pm\sqrt{-\omega_n^2} = \pm\omega_n j$$

Thus, the solution form is $\exp(j\omega_n t) = \cos(\omega_n t) + j\sin(\omega_n t)$ which are sinusoids. From Euler's formula, sine and cosine are combinations of exponentials:

$$\cos(\omega_n t) = \frac{1}{2}\left(e^{j\omega_n t} + e^{-j\omega_n t}\right) \qquad \sin(\omega_n t) = \frac{1}{2j}\left(e^{j\omega_n t} - e^{-j\omega_n t}\right)$$

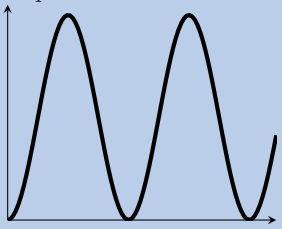
Therefore, the undamped solution is undamped sinusoids

$$y(t) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t) \tag{8.7}$$

with frequency ω_n. Thus, currents and voltages in the real world are real after all. This special case is called undamped, and since the damping coefficient σ and damping ratio ζ are zero, ω_n is the undamped natural (characteristic resonant) frequency.

Sketch of Undamped

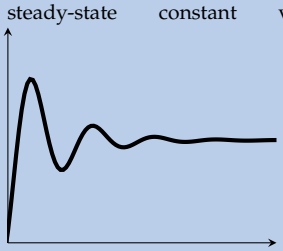
The shape of a sinusoid is a sine or cosine wave that never decays but keeps the same amplitude and oscillates forever.



Underdamped $\sigma < \omega_n$

Sketch of Underdamped

The shape is a sinusoid that exponentially decays to a steady-state constant value.



When σ is less than ω_n ($\zeta < 1$), the discriminant is negative so the roots of the characteristic equation s are complex (imaginary) roots.

$$s = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2} = -\sigma \pm j\omega_d$$

where ω_d is the damped natural frequency defined as

$$\omega_d = \sqrt{\omega_n^2 - \sigma^2} \tag{8.8}$$

This case is called underdamped. The closer ζ is to one, the less oscillations before damping.

The underdamped solution is sinusoidal with damped exponentials

$$y(t) = A_1 e^{-\sigma t} \cos(\omega_d t) + A_2 e^{-\sigma t} \sin(\omega_d t) \tag{8.9}$$

Critically Damped $\sigma = \omega_n$

Sketch of Critically Damped

The shape is a hard to distiguish between underdamped and overdamped. There is only one value where the two meet as the damping coefficient is varied. It may look like exponential decay and may sometimes have small oscillations.

When σ is equal to ω_n ($\zeta = 1$), the discriminant is zero so the roots of the characteristic equation s coincide and are thus repeated roots.

$$s = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2} = -\sigma$$

This case is called critically damped.

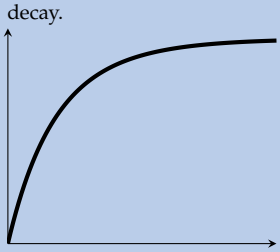
The solution cannot just be an addition of the same exponential since it simplifies to a single exponential, which is for first order. The second exponential must be a different type so the critically damped solution is

$$y(t) = A_1 e^{-\sigma t} + A_2 t e^{-\sigma t} \tag{8.10}$$

Overdamped $\sigma > \omega_n$

Sketch of Overdamped

The shape is of two exponentials added and looks like exponential decay.



When σ is greater than ω_n ($\zeta > 1$), the discriminant is positive so the roots of the characteristic equation s are distinct real roots.

$$s = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2} = s_1, s_2$$

This case is called overdamped.

The overdamped solution is the sum of two exponentials

$$y(t) = A_1 e^{-s_1 t} + A_2 e^{-s_2 t} \tag{8.11}$$

8.1.5 Step 5: Transform Back to $x = y + B/a_0$

This step remains the same regardless of system order. With the homogeneous solution now known, transform back from y to x, a voltage or current. Do not apply the ICs before transforming back as y is merely a placeholder variable and has no physical meaning.

8.1.6 Step 6: Apply Initial Conditions

The last step is to apply initial conditions (ICs) to solve for the unknown constants. Second order differential equations have two unknown constants A_1 and A_2, and thus two ICs are needed.

One IC is found the same way as first order at $x(t = 0^+)$. The second IC needed is also physically related to the circuit. Given that elemental relations for capacitors and inductors have derivatives, the second IC is $dx(t = 0^+)/dt$. Do not be tempted to take the derivative of $x(t = 0^+)$ directly, as it is a single value. Instead, use the elemental relationships for capacitors and inductors rearranged as

$$\frac{dv_c(t = 0^+)}{dt} = \frac{i_c(t = 0^+)}{C} \qquad \frac{di_L(t = 0^+)}{dt} = \frac{v_L(t = 0^+)}{L}$$

along with the continuous voltage property of capacitors $v_C(0^-) = v_C(0^+)$ and the continuous current property of inductors $i_L(t = 0^-) = i_L(t = 0^+)$ used for finding the IC $x(t = 0^+)$ to find a value for $dx(t = 0^+)/dt$.

8.2 RLC Series Example

Let's take Example 7.1.3, again find a differential equation, but then solve it too. The circuit is reproduced in Figure 8.1. Let's find the current and voltage of the capacitor over time due to the perturbation of the switch closing at time $t = 0$ by following the steps to solve circuit differential equations.

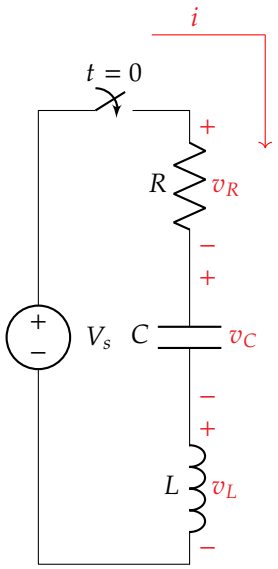


Figure 8.1: RLC Series Circuit Example

1. Use KCL&KVL and elemental relations to find a differential equation with a single unknown.

$$\text{KCL: } i_R = i_C = i_L = i$$
$$\text{KVL: } V_s = v_R + v_C + v_L$$

The elemental relations for the resistor and inductor can be used in the KVL equation without rearranging to yield

$$V_s = iR + v_C + L\frac{di}{dt}$$

Since the current is the same in series, the elemental relation for the capacitor can be used without rearranging for current such that

$$V_s = RC\frac{dv_C}{dt} + v_C + LC\frac{d^2v_c}{dt^2}$$

is an equation with a single unknown v_C.

2. To put into general form, the coefficient of the second order derivative term must be one. Reordering terms and dividing by LC gives

$$\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{v_c}{LC} = \frac{V_s}{LC}$$

Numbers as Variables

Don't forget you may like using numerical variables for known quantities to help remind which variables are known. Don't combine them though!

3. Transform the inhomogeneous equation for the unknown $x(t) = v_C(t)$ to a homogeneous equation.

$$\begin{aligned} y(t) = x(t) - \frac{B}{a_0} &= x - \frac{\frac{V_s}{LC}}{\frac{1}{LC}} = x - V_s \\ x &= y + V_s \\ \frac{dx}{dt} &= \frac{dy}{dt} \\ \frac{d^2x}{dt^2} &= \frac{d^2y}{dt^2} \end{aligned}$$

where x is v_C and plugged into the differential equation yields

$$\begin{aligned} \frac{d^2y}{dt^2} + \frac{R}{L}\frac{dy}{dt} + \frac{y + V_s}{LC} &= \frac{V_s}{LC} \\ \frac{d^2y}{dt^2} + \frac{R}{L}\frac{dy}{dt} + \frac{y}{LC} &= 0 \end{aligned}$$

which is now homogeneous.

4. The solution to a homogeneous linear constant coefficient differential equation is one of the three common types that satisfies the characteristic equation

$$\begin{aligned} s^2 + 2\sigma s + \omega_n^2 &= 0 \\ s^2 + \frac{R}{L}s + \frac{1}{LC} &= 0 \end{aligned}$$

where $\sigma = R/2L$ and $\omega_n^2 = 1/LC$ can be compared to determine which of the three solution types prevail. The numerical values for R, L, and C must be known to determine the solution type and its form. Let's call that solution form y_{trans} as its the homogeneous transient solution. y_{trans} will be one of Equation 8.9, 8.10, or 8.11.

5. Before applying the IC to find the unknown constants A_1 and A_2, transform back to unknown with physical meaning

$$\begin{aligned} x &= y + \frac{B}{a_0} \\ x &= y + V_s \\ x(t) &= y_{trans} + V_s \\ v_C(t) &= y_{trans} + V_s \end{aligned}$$

where y_{trans} is now known but has two unknown constants A_1 and A_2 as seen in Equations 8.9, 8.10, and 8.11.

6. Since the system is second order, two constants are unknown, and thus two initial conditions are needed. For the unknown capacitor voltage v_C, the ICs $v_C(t = 0^+)$ and $\frac{dv_C(t=0^+)}{dt}$ values are needed.

   Before the switch is flipped, the circuit is under dc conditions so a capacitor is treated as an open circuit and an inductor as a short circuit as shown in Figure 8.2.

   Since there is no given initial voltage of the capacitor nor does the

circuit show the capacitor connected to a voltage source that could charge it up, it is assumed that its initial voltage is zero before the switch is closed ($v_C(t = 0^-) = 0$). Using the property of a capacitor that voltage cannot abruptly change yields the IC

$$v_C(0^+) = v_C(0^-) = 0$$

right when the switch closes. To find the second IC, the current of the capacitor is needed for finding

$$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C}$$

Elements in series have the same current so $i_L(0^+) = i_C(0^+)$. Using the property of an inductor that current cannot change abruptly yields

$$i_L(0^+) = i_L(0^-) = 0$$

where the current of the inductor before switch opens is zero because it is in series with an open circuit as shown in Figure 8.2. Therefore, the second initial condition is

$$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i_L(0^+)}{C} = 0$$

If instead the current of the inductor and its derivative were needed for ICs, only $v_L(0^+)$ is still needed. Using KVL right after the perturbation (Figure 8.1),

$$\begin{aligned} V_s &= v_R(0^+) + v_C(0^+) + v_L(0^+) \\ v_L(0^+) &= V_s - v_R(0^+) - v_C(0^+) \\ v_L(0^+) &= V_s - 0 - 0 \\ \therefore \frac{di_L(0^+)}{dt} &= \frac{v_L(0^+)}{L} = \frac{V_s}{L} \end{aligned}$$

where $v_C(0^+) = 0$ is already known and $v_R(0^+) = 0$ because its current is zero $i_R(0^+) = i_L(0^+) = 0$.

Note that inductor voltage changes abruptly as $v_L(0^-) = 0$ but $v_L(0^+) = V_s$.

The two ICs are applied to

$$\begin{aligned} v_C(t) &= y_{trans} + V_s \\ \frac{dv_C(t)}{dt} &= \frac{dy_{trans}}{dt} \end{aligned}$$

where y_{trans} is known but has two unknown constants A_1 and A_2 and the derivative of y_{trans} needs to be taken. Solving these two equations determines the two unknown constants A_1 and A_2.

To sketch the response, the solution type of y_{trans} determines the shape of the transient. The first initial condition $x(0^+)$ is where the shape starts and the final steady-state value $x(t \to \infty)$ is the value the transient approaches.

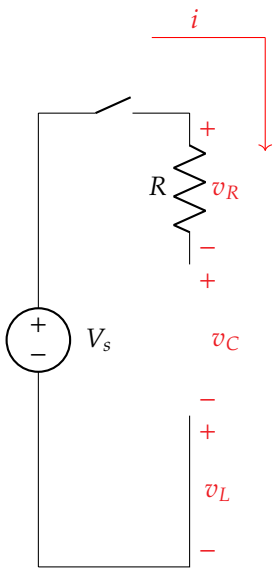


Figure 8.2: RLC series circuit before switch is flipped is under dc conditions.

Math to Review

For these problems, refresh how to take derivatives of trig functions and exponentials as well as the product and chain rules.

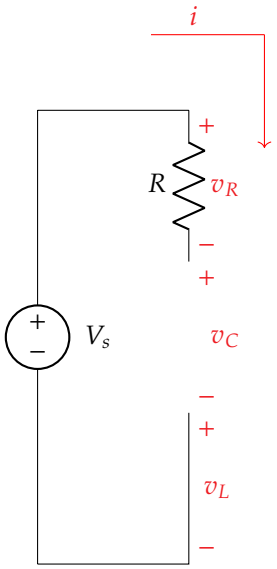


Figure 8.3: RLC series circuit long after switch is flipped and is under dc conditions.

To find the final steady-state value, analyze the circuit under dc conditions assuming the switch has been closed for a long time as shown in Figure 8.3 or find $v_C(t \to \infty)$ from the solution equation.

Under dc conditions after the switch has been closed, the inductor voltage is zero as it acts like a short circuit, but no current flows as the capacitor acts like an open circuit and all elements are in series.

Thus, with $v_L(\infty) = 0$ and $i(\infty) = 0$, the remaining voltages are

$$v_R(\infty) = i(\infty) = 0$$
$$v_C(\infty) = V_s - v_R(\infty) - v_L(\infty) = V_s$$

where (∞) means evaluated as time approaches infinity.

With $v_C(t)$ now known, all other voltages and currents can be found. Using KCL right after the perturbation,

$$i_R(t) = i_L(t) = i_C(t) = C\frac{dv_C(t)}{dt}$$

then using Ohm's Law the yields $v_R(t) = i_R(t)R$. The remaining voltage $v_L(t)$ is found using KVL right after the perturbation as

$$v_L(t) = V_s - v_R(t) - v_C(t)$$

or

$$v_L(t) = L\frac{di_L(t)}{dt}$$

8.3 Second Order Shortcut and Solving Second Order Recap

Second Order Shortcut

The final solution is a summation of the transient solution and the final steady-state value. Thus a shortcut is

$$x(t) = y_{trans} + x_{ss}$$

where $x_{ss} = x(t \to \infty)$ or $x(\infty)$. Once the solution form type is known then the shortcut equation can be used with the ICs to find the two unknown constants.

This shortcut just bypasses formally doing the transform steps. The differential equation and/or characteristic equation is still needed to find the attenuation or exponential decay constant σ, the undamped natural frequency ω_n, and thus the solution type and its corresponding form for y_{trans}.

To take the shortcut,

1. Solve for initial conditions $x(t = 0^+)$ and $dx(t = 0^+)/dt$ and final (steady-state) value $x(\infty)$.
2. Find σ and ω_n to determine solution type and transient response y_{trans} using Table 8.1
3. Apply ICs to find total response $x(t) = y_{trans} + x_{ss}$.

Table 8.1: Second Order Solution Types

Overdamped	$\sigma > \omega_n$ $\zeta > 1$	Negative real roots, decaying exponential	$y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$\sigma = \omega_n$ $\zeta = 1$	Repeated real roots	$y(t) = A_1 e^{-\sigma t} + A_2 t e^{-\sigma t}$
Underdamped	$\sigma < \omega_n$ $\zeta < 1$	Complex roots	$y(t) = A_1 e^{-\sigma t} \cos(\omega_d t) + A_2 e^{-\sigma t} \sin(\omega_d t)$

Second Order Steps Recap

In short summary, the steps to solve circuit differential equations to find the transient response are

1. Formulate the differential equation using Kirchhoff's Laws and circuit element relationships.
2. Write differential equation in normalized general form

$$\frac{d^2x}{dt^2} + 2\sigma\frac{dx}{dt} = \omega_n^2 x = B$$

3. Transform to $y = x - B/a_0$ if $B \neq 0$ and/or via $t = t - t_0$ if perturbation is not at $t = 0$.
4. Use general solution form to find σ and ω_n (and ζ) to determine solution type and transient solution form y_{trans} satisfying the characteristic equation

$$s^2 + 2\sigma s + \omega_n^2 = 0$$
$$s = -\sigma \pm \sqrt{\sigma^2 - \omega_n^2}$$

   as determine in Table 8.1 where $\zeta = \sigma/\omega_n$ and $\omega_d = \sqrt{\omega_n^2 - \sigma^2}$.
5. Transform back to $x = y + B/a_0$
6. Determine and apply ICs $x(t = 0^+)$ and $dx(t = 0^+)/dt$.

To find initial conditions, remember voltage is continuous $v_C(0^-) = v_C(0^+)$ for capacitors and current is continuous $i_L(0^-) = i_L(0^+)$ for inductors. Thus can find $t = 0^-$ values treating circuit under dc steady-state before the perturbation to help find initial conditions at $x(t = 0^+)$ and $dx(t = 0^+)/dt$. Kirchhoff's Laws and the elemental relationships are useful especially for finding $dx(t = 0^+)/dt$. Take care to use appropriate $t = 0^-$ or $t = 0^+$ circuit; both circuits' KVL&KCL equations may be needed.

8.4 Learning Targets

By combining Kirchhoff's Laws with differential equation solving techniques from mathematics, you can

- formulate circuit differential equations using KVL and KCL,
- list and apply the steps to solve (2nd order) circuit differential equations
- determine currents and voltages in 2nd order circuits (transient circuit response) for all four types, and
- draw/sketch graphical solutions to transients.

8.5 Nonelectrical Second Order Systems

The transient behavior of second order circuits is generalizable to other systems that are second order. Some examples include mechanical translating and rotating systems and electromechanical systems.

Mechanical Translating System

Mechanical systems consist of springs, dampers, and masses. A spring is modeled using Hooke's Law

$$F = -kx \tag{8.12}$$

where F is the force that drives displacement x of the spring, and the spring constant k is a measure of its stiffness. A damper provides viscous friction or damping and is modeled as

$$F = -bv = -b\frac{dx}{dt}$$

where v is velocity and b is the viscous-friction coefficient. A mass is modeled using Newton's second law of motion

$$F = ma = m\frac{d^2x}{dt^2} \tag{8.13}$$

where m is mass and a is acceleration.

Using these equations, mechanical systems can be analyzed similar to nodal analysis. For each mass (or massless node), the force balance equation is the summation of forces at the mass. The direction of each force is drawn to counteract the mass moving in an assumed positive position direction. Any displacement positions or velocities are in reference to the position on the other side of the element from the mass and are subtracted. This means no sign is included in the force equations as the proper sign comes from the displacement difference between the position or velocity at the mass minus the position or velocity on the other side of the spring or damper.

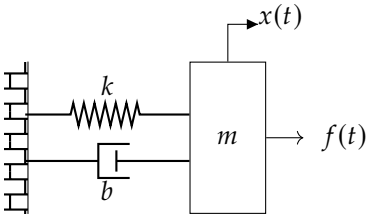


Figure 8.4: Simple Translational Mass Damper Spring System

The equation for the mass in the simple translating mechanical system in Figure 8.4 is

$$\begin{aligned} f &= F_m + F_b + F_k \\ &= m\tfrac{d^2x}{dt^2} + b\left(\tfrac{dx}{dt} - 0\right) + k(x - 0) \\ &= m\tfrac{d^2x}{dt^2} + b\tfrac{dx}{dt} + kx \end{aligned}$$

where x is the position of the mass and the position on the other side of the spring and damper elements is zero since the wall does not move.

This differential equation is the same as the inhomogeneous RLC series circuit differential equation. Thus each system shares the same transient response types (overdamped, critically damped, underdamped, or if $b = 0$ undamped). In fact, there is a voltage-force analog where a voltage source and force are the analogous driving mechanisms. There is also

a current-force analog. Table 8.2 lists the analogs for both voltage-force and current-force.

Mechanical Rotating System

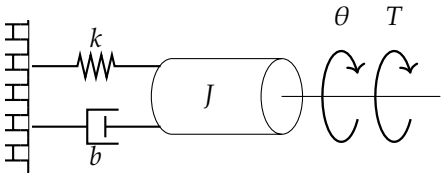


Figure 8.5: Simple Rotational Mass Damper Spring System

Mechanical rotating systems are modeled like translating mechanical systems except that the position is an angular position θ and all elements rotate due to the driving force torque T. The rotational analogs to translating systems for each element are the moment of inertia J (for mass), coefficient of rotational viscous friction b, and rotational spring constant k. The derivative of angular position is angular velocity ω, and its derivative is angular acceleration α. The torque force balance for Figure 8.5 is

$$\begin{aligned} T &= T_m + T_b + T_k \\ &= J\frac{d^2\theta}{dt^2} + b\left(\frac{d\theta}{dt} - 0\right) + k(\theta - 0) \\ &= J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + k\theta \end{aligned}$$

where θ is the position of the mass and the position on the other side of the spring and damper elements is zero since the wall does not move.

This differential equation is the same as the inhomogeneous RLC series circuit differential equation. Thus each system shares the same transient response types (overdamped, critically damped, underdamped, or if $b = 0$ undamped).

Electromechanical System

For an armature circuit with a voltage source powering a motor, the armature has a resistance and inductance. Applying KVL and assuming the armature inductance is small compared to armature resistance and can be ignored yields

$$e_a = R_a i + v_b = R_a i + K_b \frac{d\theta_m}{dt} \tag{8.14}$$

where e_a is the voltage applied, R_a is the armature resistance, i is current, v_b is the back emf (voltage), K_b is the back emf constant or electric constant also denoted as K_e (inverse of motor velocity constant K_v), and θ_m is the angular position of the motor. The torque generated by the motor is proportional to the current

$$T_m = K_t i \tag{8.15}$$

where K_t is the motor torque constant, and the motor torque T_m is related to the motor's angular position by the force balance

$$T_m = J_m \frac{d^2\theta_m}{dt^2} + D_m \frac{d\theta_m}{dt} \tag{8.16}$$

where J_m is the equivalent moment of inertia on the motor shaft and D_m is the equivalent rotational viscous friction damping coefficient on the

Table 8.2: Electrical Mechanical Analogs

	Mechanical Analog	
Electrical	Voltage Source-Force	Current Source-Force
Current, i	Velocity, v	Force, f
Voltage, $e = iR$ Resistance R	Force, $f = bv$ $R = b$	Velocity $v = \frac{1}{b}f$ $R = \frac{1}{b}$
Inductance $e = L\frac{di}{dt}$	$f = m\frac{dv}{dt} = ma$ $L = m$	$v = \frac{1}{k}\frac{df}{dt}$ $L = \frac{1}{k}$
Capacitance, $i = C\frac{de}{dt}$	$v = \frac{1}{k}\frac{df}{dt}$ $C = \frac{1}{k}$	$f = m\frac{dv}{dt} = ma$ $C = m$
$P = ie$ $\sum_{loop} v = 0$ $\sum_{node} i = 0$	$P = fv$ $\sum f = 0$	$P = vf$ $\sum f = 0$

motor shaft. Putting Equations 8.14, 8.15, and 8.16 together, the system is modeled as

$$e_a = \frac{R_a}{K_t}\left(J_m\frac{d^2\theta_m}{dt^2} + D_m\frac{d\theta_m}{dt}\right) + K_b\frac{d\theta_m}{dt} \tag{8.17}$$

which is a second order system. Thus it shares the same transient response types of second order systems (overdamped, critically damped, underdamped, or if $D_m = 0$ undamped).

AC Circuits

AC Circuits 9

When introducing alternating current (ac) sources, the steady-state response is no longer constant but continuously varies with time. AC source circuits can be analyzed using differential equation techniques. Additionally, voltages and currents can be represented as phasors (complex numbers) allowing analysis through algebra, albeit messy algebra, rather than calculus.

While reading the chapter focus on learning how to

- describe, graph, and calculate sinusoidal amplitude, phase (shifts), period, and frequency.
- define phasor and express it in rectangular, polar, and exponential form.
- convert phasor between rectangular, polar, and exponential form.
- convert sinusoids and their integrals and derivatives from time domain to phasor/complex frequency domain.
- define impedance and express voltage and current of resistors, inductors, and capacitors using impedance.
- determine inductor and capacitor behavior at high and low frequencies.
- calculate equivalent impedance for resistors, inductors, and capacitors.
- apply Ohm's Law and Kirchhoff's Laws to sinusoidal circuits using phasors and impedance.
- use phasors and impedances and dc circuit analysis tools to analyze ac circuits.

and if needed use differential equation techniques for finding a particular solution for ac transients.

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9.1 Driven Circuits

Chapter 7 on First Order Circuits and Chapter 8 on Second Order Circuits use knowledge from differential equations to find the transient response due to a perturbation from a constant dc source. Circuits may instead be driven with time varying sources. If the driving source is $f(t)$ then the differential equation in general form is

$$a_0 x + a_1 \frac{dx}{dt} + a_2 \frac{d^2x}{dt^2} + \ldots = f(t) \tag{9.1}$$

To solve circuits with time varying sources, one uses

$$x(t) = x_p + x_c \tag{9.2}$$

where x_p is a particular solution and x_c is the complementary or homogeneous solution.

The complementary or homogeneous solution is found as before for first and second order transients using the homogeneous differential equation. The key is finding the particular solution, which depends on the form of the driving function $f(t)$, using mathematical techniques for solving differential equations.

For ac circuits, the typical driving function is a sinusoid. If $f(t) = \sin(\omega t)$ then there is a particular solution that is sinusoidal

$$x_p(t) = A \sin\left(\omega t + \phi\right) \tag{9.3}$$

where ϕ is a phase difference and ω is the driving function frequency that does not have to be related to the undamped natural frequency ω_n or damped frequency ω_d.

Alternatively, the particular solution may be expressed as

$$x_p(t) = A_1 \sin \omega t + A_2 \cos \omega t$$

where the coefficients are solved for by putting the particular solution into the differential equation.

9.1.1 First Order Example

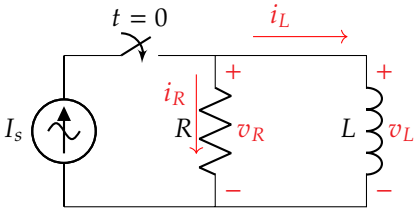

Figure 9.1: RL Parallel AC Source Driven Circuit

Take Example 7.1.2 of a RL parallel circuit reproduced with an ac current source instead of a dc current source in Figure 9.1. The differential equation is derived as follows starting with KCL

$$\begin{aligned} I_s(t) &= i_R + i_L \\ I_s(t) &= \frac{v}{R} + i_L \\ I_s(t) &= \frac{L}{R}\frac{di_L}{dt} + i_L \end{aligned}$$

where KVL is used to equate the resistor and inductor voltages in order to apply the inductor elemental relationship to get the single unknown i_L.

If $I_s(t) = \sin \omega t$ then the particular solution type is

$$i_p(t) = A_1 \cos \omega t + A_2 \sin \omega t$$

and substituting the particular solution into the differential equation for i_L will yield the two unknown constants.

$$\begin{aligned}
I_s(t) &= \frac{L}{R}\frac{di_L}{dt} + i_L \\
\sin \omega t &= \frac{L}{R}\left(-A_1\omega \sin \omega t + A_2\omega \cos \omega t\right) + A_1 \cos \omega t + A_2 \sin \omega t \\
\sin \omega t &= \left(A_2 - A_1\omega\frac{L}{R}\right)\sin \omega t + \left(A_1 + A_2\omega\frac{L}{R}\right)\cos \omega t \\
A_1 + A_2\omega\frac{L}{R} &= 0 \quad \rightarrow A_1 = -A_2\omega\frac{L}{R} \\
A_2 - A_1\omega\frac{L}{R} &= 1 \\
A_2 + A_2\left(\omega\frac{L}{R}\right)^2 &= 1 \quad \rightarrow A_2 = \left[1 + \left(\omega\frac{L}{R}\right)^2\right]^{-1} \\
&\qquad \rightarrow A_1 = -\omega\frac{L}{R}\left[1 + \left(\omega\frac{L}{R}\right)^2\right]^{-1}
\end{aligned}$$

This yields a particular solution of

$$i_p(t) = \frac{-\omega\frac{L}{R}\cos \omega t + \sin \omega t}{1 + \left(\omega\frac{L}{R}\right)^2}$$

Using the trig identity $a \cos t + b \sin t = \sqrt{a^2 + b^2}\sin\left(t + \tan^{-1}(a/b)\right)$, the particular solution can be written as a sinusoid with a phase difference

$$i_p(t) = \frac{\sin(\omega t + \phi)}{\sqrt{1 + \left(\omega\frac{L}{R}\right)^2}}, \quad \phi = \tan^{-1}\left(-\omega\frac{L}{R}\right)$$

This same process can be used for second order circuits though the math becomes messier.

Complex algebra and Laplace transforms offer a more efficient alternative to differential equation techniques. To that end, phasors represent sinusoids as complex numbers, and Laplace transforms convert differential equations into algebraic equations in the complex frequency domain.

9.2 Sinusoids and Phasors

For steady-state ac responses, phasors convert sinusoid signals (voltages and currents) from the time domain to the complex frequency domain (s-domain). Phasors are complex numbers and can be thought of as special case of Laplace transforms, replacing differential calculus with complex algebra.

In ac circuits, the driving force is a sinusoid (sine or cosine), which is characterized by an amplitude A, angular frequency ω, and phase ϕ as

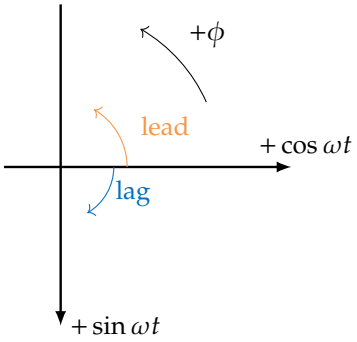

Figure 9.2: Sinusoids on Complex Plane

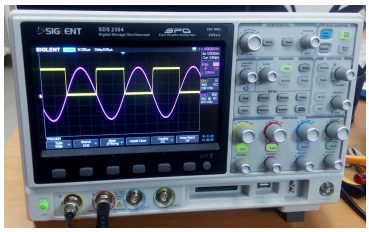

Figure 9.3: Oscilloscope
Wild Pancake, CC BY 4.0 via Wikimedia Commons

depicted in Figure 9.4. When comparing two sinusoids, a sinusoid leads the other when its maximum comes first and the other sinusoid lags. The cosine curve (blue compound-style line) is a phase-shifted sine curve that leads the sine curve (orange solid line) by 90°.

Sinusoids are represented graphically on a complex plane (s-domain) as shown in Figure 9.2. This graphical representation with +sin in the negative vertical axis and +cos in positive horizontal axis is very useful to remember lead, lag, and phase shift ϕ when comparing sinusoids. A positive phase shift moves the sinusoid clockwise in the complex plane. For example, sine has a positive 90°phase shift to represent cosine. Therefore, cosine is 90°clockwise from sine and leads sine by 90°(sine lags cosine).

It is best to measure these signals with an oscilloscope to see the signal over time. Note the relationships between amplitude and peak-to-peak amplitude as well as frequency f, angular frequency ω, and period T. The mathematical representation of a sinusoid uses angular frequency and amplitude while the oscilloscope typically provides peak-to-peak amplitude and frequency. An example screenshot on an oscilloscope is shown in Figure 9.3.

A multimeter can also measure the voltage or current of a sinusoid; however it provides only a single value, the root means squared (RMS) value, which is an effective dc equivalent of the ac voltage or current. Mathematically it is

$$x_{RMS} = \sqrt{\frac{1}{T}\int_0^T x^2(t)dt}$$

which relates to amplitude as $x_{RMS} = A/\sqrt{2}$ for a sinusoid.

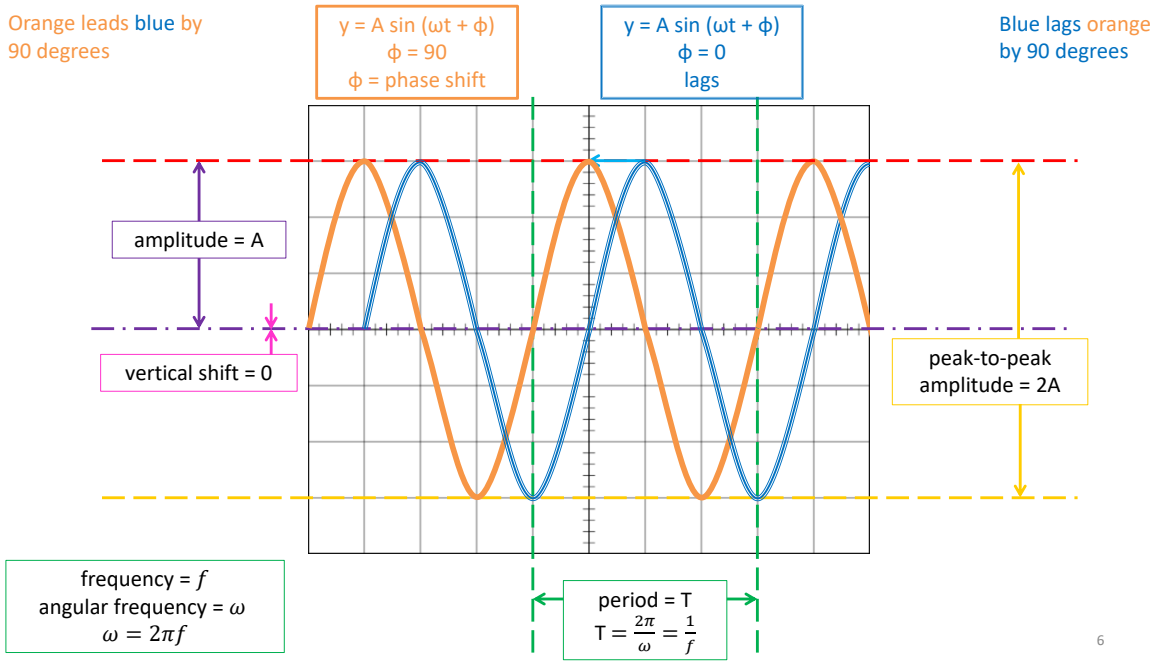

Figure 9.4: Sinusoids

To analyze ac circuits with algebra, sinusoid signals are represented as phasors, that can be added, subtracted, multiplied, divided, differentiated, and integrated. Thus, the elemental relationships can be expressed with phasors, which introduces the concept of impedance.

9.2.1 Phasors and Complex Numbers

A phasor is a complex number representing a sinusoid. Complex numbers can be expressed in rectangular, polar, or exponential form. A complex number z is represented as

$$\begin{array}{ll} \text{Rectangular} & z = x + jy \\ \text{Polar} & z = r\angle\phi = r\underline{/\phi} \\ \text{Exponential} & z = re^{j\phi} \end{array}$$

and can be represented graphically on the complex (frequency) plane where the horizontal axis is the real axis and the vertical axis is the imaginary axis as shown in Figure 9.5.

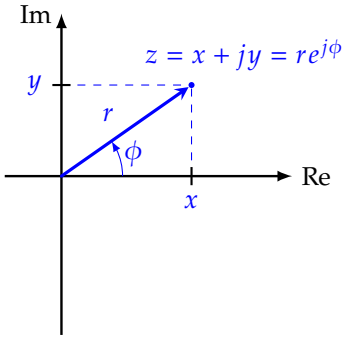

Figure 9.5: Phasor Graphed on Complex Plane

Using complex numbers to represent sinusoids simplifies calculations. To add and subtract complex numbers, rectangular form is best:

$$z_1 \pm z_2 = x_1 \pm x_2 + j\,(y_1 \pm y_2)$$

and polar form is best to multiply and divide:

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \underline{/\phi_1 + \phi_2} \\ \frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{/\phi_1 - \phi_2)} \quad \frac{1}{z} &= \frac{1}{r}\underline{/-\phi} \\ \sqrt{z} &= \sqrt{r}\underline{/\phi/2} \end{aligned}$$

In order to switch to a different form, the following geometric relations seen in Figure 9.5 convert between different representations:

$$\begin{aligned} r = \sqrt{x^2 + y^2} \qquad & \phi = \tan^{-1}\frac{y}{x} \\ x = r\cos\phi \qquad & y = r\sin\phi \\ z = r\left(\cos\phi + j\sin\phi\right) & \end{aligned}$$

Also, recall the complex conjugate of a complex number z is

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi}$$

and the imaginary number j is really just the imaginary part of complex number so its inverse is its complex conjugate:

$$\begin{aligned} j = 0 + j \rightarrow \phi = \tan^{-1}\frac{1}{0} = \tan^{-1}\infty = 90^\circ = \angle 90^\circ \\ \frac{1}{j} = -j = \underline{/-90^\circ} \end{aligned}$$

All these relationships can be derived from Figure 9.5 and are algebraic. Thereby, expressing sinusoids as phasors, avoids solving a differential equation for steady-state ac responses. Let's see how a sinusoid is represented as a phasor.

9.2.2 Sinusoid as a Phasor

For a cosine sinusoid signal

$$v(t) = V_m \cos(\omega t + \phi)$$

the real part from Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$ is cosine so the sinusoid can be written as

$$v(t) = Re[V_m e^{j(\omega t+\phi)}] = Re[V_m e^{j\phi} e^{j\omega t}] = Re[\bar{V} e^{j\omega t}]$$

where $\bar{V}$ is a complex number representing the cosine sinusoid as

$$\bar{V} = V_m e^{j\phi} = V_m \angle\phi$$

which converts the sinusoid from the time domain to the complex frequency domain. Note the $e^{j\omega t}$ is left out of the notation but still applies! The phasor still maintains its specific angular frequency.

Remark 9.2.1 A phasor representing a sinusoid has an implied angular frequency that cannot be forgotten. Phasors as complex numbers cannot be combined unless they have the same angular frequency!

If a signal is a sine then a phase shift of -90° is needed first since sine lags cosine by 90° as discerned from Figure 9.2.

Writing a sinusoid as a phasor converts the signal from the time domain to the phasor domain (complex frequency domain) where $e^{j\omega t}$ is suppressed notationally. The angular frequency is constant but the response depends on it!

Some signals are differentials (or integrations), which can be represented in the complex frequency domain too. Recall Laplace transform properties for derivatives and integrals

$$\begin{aligned} \text{First derivative} \quad & \frac{df}{dt} \overset{\mathscr{L}}{\longleftrightarrow} sF(s) - \cancelto{0}{f(0^-)} \\ \text{Integral} \quad & \int_0^t f(x)dx \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s}F(s) \end{aligned}$$

where for phasors the initial condition is assumed to be zero and $s = j\omega$. For the characteristic equation for second order, the roots are $s = -\sigma \pm j\omega$, but if there is no damping ($\sigma = 0$), they become purely sinusoidal like we have here in steady-state ac circuits! Therefore, differentiating and integrating phasors representing sinusoids in steady-state comes from Laplace transforms. For a sinusoidal voltage v these are

$$\begin{aligned} \text{Differentiating} \quad & \frac{dv}{dt} \overset{\mathscr{L}}{\longrightarrow} s\bar{V} = j\omega\bar{V} = j\omega V_m \angle\phi \\ \text{Integrating} \quad & \int v dt \overset{\mathscr{L}}{\longrightarrow} \frac{\bar{V}}{s} = \frac{\bar{V}}{j\omega} = \frac{V_m\angle\phi}{j\omega} \end{aligned}$$

where $\bar{V}$ is complex with real and imaginary parts and a constant frequency ω so $\bar{V}$ does not depend on time t. This is a special case of Laplace transform for steady-state ac circuits. More generally, Laplace transforms can solve differential equations algebraically, which is useful for solving transient circuits and other non-electrical dynamic systems like those described in Sections 7.5 and 8.5.

9.2.3 Elemental Relationships with Phasors

The elemental relationships for resistors, inductors, and capacitors apply to phasors too.

Resistors If the current through a resistor is $i_R = I_m \cos(\omega t + \phi)$ then it can be multiplied by resistance, which is a constant, simply as

$$\begin{aligned} v_R = i_R R &= R I_m \cos(\omega t + \phi) \\ \bar{V}_R &= R I_m \angle \phi \\ \bar{V}_R &= R \bar{I}_R \end{aligned}$$

which gives voltage. Therefore, Ohm's Law applies for phasors!

Inductors For an inductor with current $i_L = I_m \cos(\omega t + \phi)$, its derivative is needed for the elemental relationship such that

$$\begin{aligned} v_L = L\frac{di_L}{dt} &= L\frac{dI_m \cos(\omega t + \phi)}{dt} \\ &= -\omega L I_m \sin(\omega t + \phi) \\ &= \omega L I_m \cos(\omega t + \phi + 90^\circ) \\ \bar{V}_L &= \omega L I_m \underline{/\phi + 90^\circ} \\ &= j\omega L I_m \underline{/\phi} \\ \bar{V}_L &= j\omega L \bar{I}_L \end{aligned}$$

which is the same result if started with current as a phasor and differentiated it.

Capacitors For capacitors, the elemental relationship with phasors is

$$\begin{aligned} i_c &= C\frac{dv}{dt} \\ \bar{I}_C &= j\omega C \bar{V}_C \rightarrow \bar{V}_C = \frac{\bar{I}_C}{j\omega C} \end{aligned}$$

9.2.4 Impedance

Impedance (Z) is the total opposition to current in an ac circuit. Quantitatively, impedance is defined as the ratio of voltage to current. Rearranging Ohm's Law for resistance yields

$$R = \frac{\bar{V}}{\bar{I}} := Z_R \tag{9.4}$$

which means impedance has the same units as resistance (Ohms).

It is not a phasor because it is not a sinusoid, but it is a complex number that is frequency dependent with a real part R and imaginary part X . Given the definition of impedance, inductor and capacitor impedance, also with units of Ohms, is defined as

$$Z_L = j\omega L \tag{9.5}$$

$$Z_C = \frac{1}{j\omega C} \tag{9.6}$$

Impedance Z and Admittance Y

Impedance

$$Z = R \pm jX$$

where R is the real part called resistance and X is the imaginary part called reactance, represents opposition to current flow through inductors and capacitors. Where impedance is defined as the voltage to current ratio, the admittance is its inverse defined as current to voltage ratio

$$Y = G + jB$$

where G is the real part called conductance and B is the imaginary part called susceptance representing the ease of current flow through reactive components like capacitors and inductors. In parallel ac circuits admittances can be added directly.

For dc circuits $\omega = 0$, so $Z_L = 0$ and $Z_C = \infty$ means inductors act like short circuits and capacitors act like open circuits under dc conditions. At really high frequencies when $\omega \to \infty$, $Z_L = \infty$ and $Z_C = 0$ so inductors act like open circuits and capacitors like short circuits.

Since impedance is defined as the voltage-to-current ratio, all elements behave like resistors and can be treated as resistors with an impedance for analysis.

9.3 AC Circuit Analysis

The elemental relationships for resistors, inductors, and capacitors all hold for ac circuits. What about other circuit analysis tools?

9.3.1 Equivalent Impedance

Elements in series or parallel can be combined into an equivalent impedance. For series elements, their impedances sum directly. For parallel elements, the inverse of equivalent impedance is the sum of the inverse impedances just like resistance for resistors.

$$Z_{series} = \sum_n Z_n \tag{9.7}$$

$$Z_{parallel} = \left[\sum_n \frac{1}{Z_n} \right]^{-1} \tag{9.8}$$

9.3.2 Kirchhoff's Laws

Kirchhoff's Laws are based on the conservation of charge and energy. They apply in the complex frequency (phasor) domain just as they do in the time domain.

$$\text{KCL: } \sum_{in} \bar{I} = \sum_{out} \bar{I} \tag{9.9}$$

$$\text{KVL: } \sum_{loop} \bar{V} = 0 \tag{9.10}$$

This is crucial because Kirchhoff's Laws form the basis for nodal analysis, mesh analysis, source transforms, Thevenin and Norton theorems, and superposition. It also confirms equivalent impedances match equivalent resistances.

Thus, the circuit analysis toolbox used for dc circuits applies to ac circuits with two important caveats. First, since phasors have a specific angular frequency, only phasors with the same frequency may be combined in the complex frequency domain. When applying superposition for sources with different angular frequencies, the quantity of interest from each source must be combined in the time domain (not in the complex frequency domain). Second, there are different components to power for sinusoids so the maximum power transfer is not simply $p = iv$.

9.3.3 Steps to Analyze AC Circuits

Given that all elements can be treated like resistors once converted to the complex frequency domain using impedance, Kirchhoff's Laws, and other dc analysis tools apply, the steps to analyze ac circuits are

1. Convert circuit to the complex frequency domain (phasor domain).
2. Solve like a dc circuit using phasors and impedances
3. Convert back to time domain

Note that responses from sources with different frequencies cannot be combined in the complex frequency domain (only in the time domain). Therefore, for superposition combine responses after converting back to the time domain if sources have different frequencies. Additionally, the average power not the instantaneous power $p = iv$ determines maximum power transfer.

To learn how to apply the steps to analyze an ac circuit, let's look at an example circuit with two loops shown in Figure 9.6. To find the current $i_s(t)$, follow the steps:

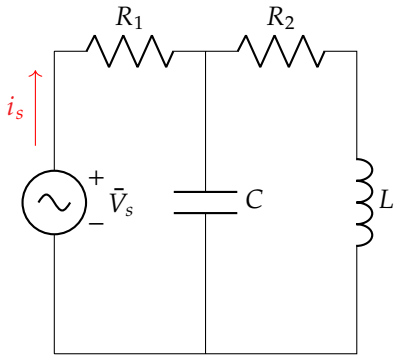

Figure 9.6: Two Loop AC Circuit Example

1. Converting to impedances yields

$$\begin{aligned} Z_1 &= R_1 \\ Z_2 &= R_2 \\ Z_C &= \frac{1}{j\omega C} \\ Z_L &= j\omega L \end{aligned}$$

   and assuming the source is $v_s(t) = V_s \cos(\omega t + \phi_v)$, its phasor representation is $\bar{V}_s = V_s \angle \phi$.
2. Often there are multiple analysis tools for finding the quantity of interest. For this circuit, equivalent impedances, nodal anlaysis, and source transforms are options. Since the desired current is through the source, equivalent impedances may prove best suited since

$$i_s(t) = \bar{I}_s = \frac{\bar{V}_s}{Z_{eq}}$$

   where the equivalent impedance is found combining Z_2 in series with Z_L and then that impedance in parallel with Z_C and finally that result in series with Z_1 as follows:

$$\begin{aligned} Z_{2L} &= Z_2 + Z_L \\ Z_{2LC} &= \frac{Z_{2L} Z_C}{Z_{2L} + Z_C} = \frac{(Z_2 + Z_L) Z_C}{Z_2 + Z_L + Z_C} \\ Z_{eq} &= Z_{2LC} + Z_1 = \frac{(Z_2 + Z_L) Z_C}{Z_2 + Z_L + Z_C} + Z_1 \end{aligned}$$

   Given numerical values for R_1, R_2, L, and C will yield a single complex number as a phasor for $\bar{I}_s = I_s \angle \phi_i$ where I_s and ϕ_i are known numerical values. The math to find them is messy and requires switching between polar form (for multiplication and division) and rectangular form (for addition and subtraction). Use tools like Octave and Matlab to save time and check work.
3. Using the phasor $\bar{I}_s$, converting back to the time domain yields $i_s(t) = I_s \cos(\omega t + \phi_i)$.

9.3.4 Power in AC Circuits

Because power is not linear and voltage and current are sinusoidal rather than constant, energy flow is described in several ways—instantaneous, apparent, reactive, and complex. Additionally, the Maximum Power Transfer for ac circuits needs to be extended for load impedance.

Instantenous power is the rate of the flow of energy at a point in a circuit and is the familiar

$$p(t) = i(t)v(t) \tag{9.11}$$

Average power or real power is the average instantaneous power over one period T

$$P = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2}V_m I_m \cos(\phi_v - \phi_i) \tag{9.12}$$

where $v(t) = V_m \cos(\omega t + \phi_v)$ and $i(t) = I_m \cos(\omega t + \phi_i)$. The phase difference determines the average power delivered to a load, which is the only useful power. Since average power no longer depends on time, it can also be calculated in the complex frequency domain using phasors and using RMS values as

$$P = \frac{1}{2}Re[\bar{V}\bar{I}^*] = V_{RMS}I_{RMS}\cos(\phi_v - \phi_i) \tag{9.13}$$

If the phase difference is zero, the circuit or load is purely resistive. If the phase difference is $\pm 90^\circ$, it is purely reactive with $P = 0$.

Apparent power is the total power flow regardless of whether it is useful power doing work. It is the magnitude of the complex power S and is defined as the product of RMS values for voltage and current measured in volts-amperes (VA)

$$|S| = V_{RMS}I_{RMS} \tag{9.14}$$

Power factor is the cosine of the phase difference between voltage and current or equivalently the cosine of the angle of load impedance. It is the dimensionless ratio of the average power to apparent power

$$\text{pf} = \frac{P}{|S|} = \cos(\phi_v - \phi_i) \tag{9.15}$$

Reactive power is a measure of energy exchange of the reactive part of a load in units of volt-ampere reactive (VAR). This power does not do useful work or get consumed by a load but oscillates between the power source and load every cycle. Related to the reactance of inductors and capacitors, reactive power is

$$Q = I_{RMS}^2 X \tag{9.16}$$

where X is reactance (imaginary part of impedance) making Q the imaginary part of complex power. When $Q = 0$, the load is purely resistive (pf=1). Capacitive loads have $Q < 0$ (leading pf), and inductive loads have $Q > 0$ (lagging pf).

Complex power is the real power and reactive power

$$S = P + jQ \tag{9.17}$$

in a single complex number.
It contains all relevant power information and can be represented as a power triangle on the complex plane as shown in Figure 9.7.
It is defined as the product of the voltage and the complex conjugate of the current which can also be expressed with RMS values as

$$S = \frac{1}{2}\bar{V}\bar{I}^* = \bar{V}_{RMS}\bar{I}^*_{RMS} = V_{RMS}I_{RMS}\underline{/\phi_v - \phi_i} \tag{9.18}$$

where $\bar{V}_{RMS} = V_{RMS}\angle\phi_v$ and $\bar{I}_{RMS} = V_{RMS}\angle\phi_i$.

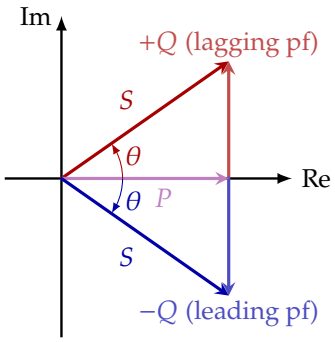

Figure 9.7: Power Triangle where $\theta = \phi_v - \phi_i$

Maximum Power Transfer

To extend the maximum power transfer theorem to ac circuits, consider a load impedance $Z_L = R_L + jX_L$. Applying Thevenin's theorem, the circuit is simplified to a Thevenin voltage $\bar{V}_{th}$ in series with a Thevenin impedance $Z_{th} = R_{th} + jX_{th}$. The current through the load is

$$\bar{I} = \frac{\bar{V}_{th}}{Z_{th} + Z_L} = \frac{\bar{V}_{th}}{(R_{th} + jX_{th}) + (R_L + jX_L)}$$

so the average power delivered to the load is

$$P = \frac{1}{2}\bar{V}\bar{I}^* = |\bar{I}|^2 R_L = \frac{|\bar{V}_{th}|^2 R_L/2}{(R_{th} + R_L)^2 + (X_{th} + X_L)^2}$$

where $|\bar{I}|^2 = \bar{I} \times \bar{I}^*$. To maximize power $\partial P/\partial R_L$ and $\partial P/\partial X_L$ are set to zero yielding the criteria

$$X_L = -X_{th}$$
$$R_L = \sqrt{R_{th}^2 + (X_{th} + X_L)^2}$$

Combining the criteria simplifies R_L criteria to

$$\begin{aligned} R_L &= \sqrt{R_{th}^2 + (X_{th} - X_{th})^2} \\ &= R_{th} \end{aligned}$$

Thus, maximum average power transfer occurs when

$$\begin{aligned} Z_L &= R_L + jX_L \\ &= R_{th} - jX_{th} \\ &= Z_{th}^* \end{aligned}$$

Therefore to transfer the maximum power, the load impedance must equal the complex conjugate of the Thevenin impedance.

$$Z_L = Z_{th}^* \tag{9.19}$$

9.4 Learning Targets

AC circuits with sinusoidal sources can be analyzed now that you can

- describe, graph, and calculate sinusoidal amplitude, phase (shifts), period, and frequency.
- define phasor and express it in rectangular, polar, and exponential form.
- convert phasor between rectangular, polar, and exponential form.
- convert sinusoids and their integrals and derivatives from time domain to phasor/complex frequency domain.
- define impedance and express voltage and current of resistors, inductors, and capacitors using impedance.
- determine inductor and capacitor behavior at high and low frequencies.
- calculate equivalent impedance for resistors, inductors, and capacitors.
- apply Ohm's Law and Kirchhoff's Laws to sinusoidal circuits using phasors and impedance.
- use phasors and impedances and dc circuit analysis tools to analyze ac circuits.

and if needed use differential equation techniques for finding a particular solution for ac transients.

Basics of Frequency Response and Filters

10

AC circuits have sinusoidal voltages and currents which are characterized by an amplitude, angular frequency, and phase. Since they are not characterized by time, they are typically analyzed in the complex frequency domain (phasors and impedance), and it is useful to graph the response at different frequencies using bode plots of the circuit transfer function. Bode plots graph amplitude and phase with respect to frequency showing at which frequencies signals are allowed and which are blocked (filtered out).

While reading the chapter focus on learning how to

- define a transfer function both mathematically and in words.
- convert transfer function gain to units of decibels to express ratios of power, voltage, or current.
- define a bode plot and interpret bode plot graphs, mainly to determine filter type.
- describe the function of filter circuits, classify them as passive or active, and determine filter type: low-pass, high-pass, band-pass, and band-stop.
- design a filter.

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10.1 Transfer Function

Laplace, Fourier, and Phasors

Laplace transforms, Fourier transforms, and phasors are all related. Laplace transforms model both transients and steady-state as characterized by attenuation and frequency and span the entire complex plane ($s = -\sigma \pm j\omega$). For steady-state, there is no damping ($\sigma = 0$). The Fourier transform models steady-state as characterized by magnitude and phase as a function of frequency on the imaginary axis of the complex plane ($s = j\omega$). Phasors are a single-frequency point ($s = j\omega$). They suppress the frequency while expressing magnitude and phase at a specific frequency.

Circuits are designed to provide a specific output for a given input. When output-to-input ratio is expressed mathematically, the circuit has a transfer function (TF) defined as

$$TF = \frac{C(s)}{R(s)} = \frac{\text{output}}{\text{input}} \tag{10.1}$$

where the C is the output, R is the input, and $s = j\omega$ for steady-state with no damping ($\sigma = 0$). The transfer function is in the complex frequency domain where the input and output are Laplace transforms with initial conditions assumed to be zero. Transfer functions can be represented as block diagrams with the transfer function inside the block and arrows indication input and output signals as shown in Figure 10.1.

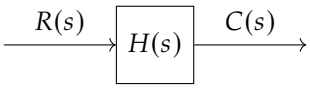


Figure 10.1: Transfer Function Block Diagram

10.2 Bode Plots

In frequency response analysis, the transfer function is graphed on a bode plot in the frequency domain with angular frequency ω on the x-axis using a logarithmic scale. The two remaining signal characteristics—amplitude (also called magnitude, gain, modulus) and phase—are each plotted on separate y-axes as shown in Figure 10.2. The magnitude is expressed in decibels (dB) on a log scale and phase in degrees on a linear scale.

The transfer function and its bode plot show how the circuit system will respond to an input at any angular frequency. The output of the system maintains the same angular frequency but its amplitude is scaled by the magnitude of the transfer function (Equation 10.2) and its phase is shifted by the argument of the transfer function (Equation 10.3).

The magnitude is found by taking the complex number of the transfer function multiplied by its complex conjugate. Geometrically, the magnitude represents the distance from the origin to the complex number of the transfer function in the complex plane as shown as r in Figure 9.5. Mathematically, for a transfer function $H(s) = z$, the magnitude of H(s) is

$$r = |H(s)| = \sqrt{z \cdot z^*} \tag{10.2}$$

The phase shift is found by taking the argument of the transfer function. Geometrically, the phase shift is the angle between the positive real axis and the line from the origin to the complex number representing transfer function in the complex plane as shown as ϕ in Figure 9.5. Mathematically, for a transfer function $H(s) = z$, the phase or argument of H(s) is

$$\phi = \arg(H(s)) = \arg(z) \tag{10.3}$$

where ϕ in rectangular form can be computed with the arctan function.

10.2.1 Decibel and Half-power Point

On bode plots, magnitude (usually voltage) typically has units of decibels. The unit **decibel**, one tenth of a bel, expresses a 10:1 ratio of two power quantities as

$$10 \log_{10} \frac{P}{P_0} \text{dB} \tag{10.4}$$

where P is the output power and P_0 is the input or reference power. This means a power ratio of 10 is 10 dB and a power ratio of one is zero dB. Typically, power is lost meaning the power ratio is less than one and decibels are negative.

Power is proportional to the square of voltage or current so to express the transfer function for voltage or current in decibels for transfer function $H(s)$

$$H = \frac{z_{out}}{z_{in}} \rightarrow 10 \log_{10} \frac{z_{out}^2}{z_{in}^2} \text{ dB} = 20 \log_{10} H \text{ dB} \tag{10.5}$$

where z_{out} is the output and z_{in} is the input voltage or current. The maximum output is when the input and output are equal

$$H = \frac{z_{out}}{z_{in}} = 1 \rightarrow 20 \log_{10} 1 = 0 \text{ dB}$$

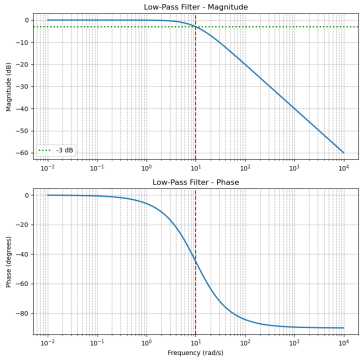


Figure 10.2: Bode Plot
Code was generated by ChatGPT (OpenAI) in response to a user query in December 2025. The model produced a Python script for plotting first order bode plots of the four filter types.

The bode plot in Figure 10.2 marks the -3dB point, which is called the **half-power point**, cutoff frequency, or corner frequency (ω_c or f_c). It is the point where the power has dropped to half the peak and corresponds to a voltage (or current) drop of $1/\sqrt{2}$ since power is proportional to its square. Beyond the cutoff frequency, signals attenuate (drop) on one side of the frequency and pass unattenuated on the other side. This behavior is a filter circuit that can be designed for a specific cutoff frequency. To design a filter for a specific cutoff frequency, set the transfer function equal to $1/\sqrt{2} \approx 0.707 \approx -3\text{dB}$ and choose circuit elements to fulfill the equality. Select resistors last as they provide more flexibility in choice and configuration. For example, the Figure 10.4b voltage divider low-pass RC filter circuit transfer function is

$$H = \frac{V_{out}}{V_{in}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1}$$

and its magnitude set to the half-power point yields the cutoff frequency

$$\begin{aligned}
|H| &= \sqrt{\left(\frac{1}{1 + j\omega RC}\right)\left(\frac{1}{1 - j\omega RC}\right)} \\
\frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} \\
2 &= 1 + \omega^2 R^2 C^2 \\
1 &= \omega^2 R^2 C^2 \\
1 &= \omega RC \\
\omega &= \frac{1}{RC} \\
f &= \frac{1}{2\pi RC}
\end{aligned}$$

10.3 Filters

Filter circuits selectively pass or block frequencies at the circuit output. There are four fundamental types of filters.

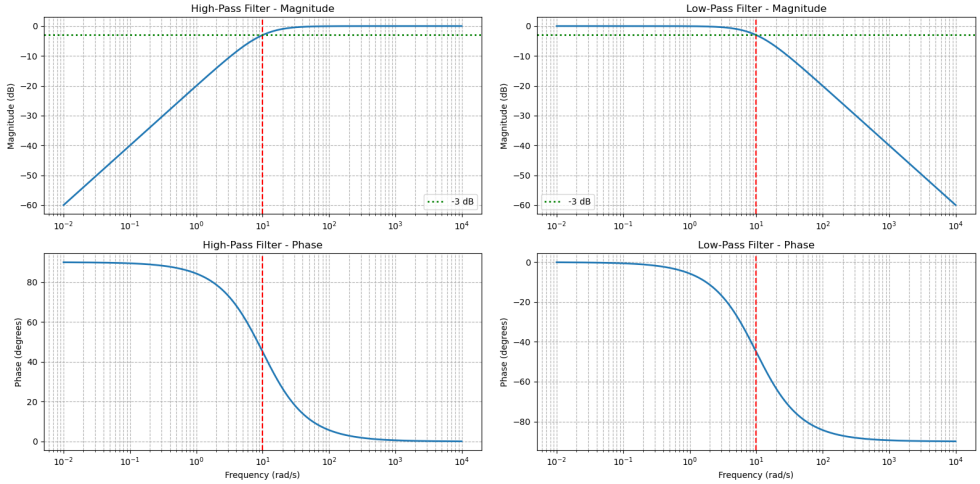
High-pass filters allow high frequencies above a cutoff frequency to pass to the output while blocking low frequencies.

Low-pass filters allow low frequencies below a cutoff frequency to pass to the output while blocking high frequencies.

Band-pass filters allow a range or band of frequencies between a low and high cutoff frequency to pass to the output while blocking frequencies below and above the band.

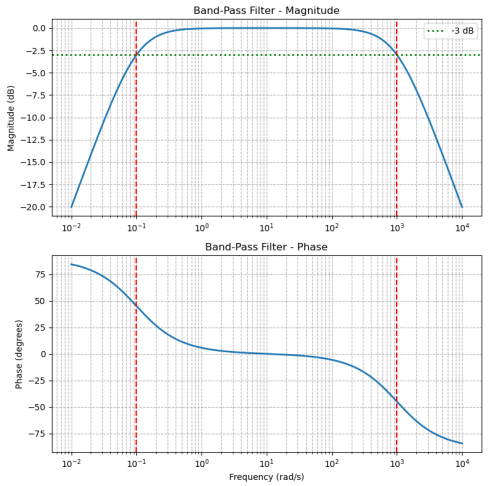
Band-stop filters (band-reject or notch filters) allow frequencies below a low cutoff and above a high cutoff frequency to pass to the output while blocking frequencies between the cutoffs.

Figure 10.3 shows the bode plot of each type of filter assuming a first order filter. Ideally, filters abruptly flip from passing and blocking (attenuating)

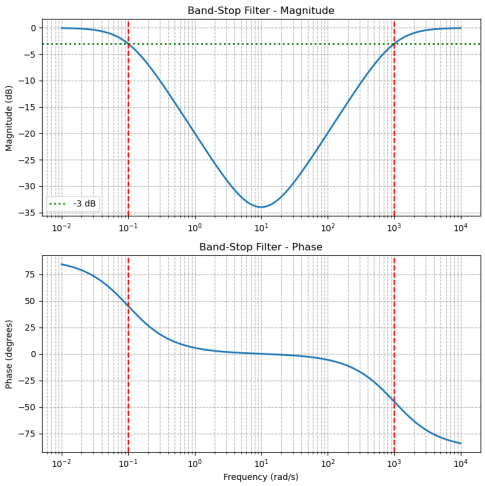


(a) High-pass Filter

(b) Low-pass Filter



(c) Bandpass Filter



(d) Bandstop Filter

Figure 10.3: Bode Plots of the Four Filter Types
Code was generated by ChatGPT (OpenAI) in response to a user query in December 2025. The model produced a Python script for plotting first order bode plots of the four filter types.

signals at the cutoffs forming a step shape bode plot; however, as the cutoff frequencies are approached, the gain gradually transitions from gain 0 dB (passing full signal) and slopes down to negative decibel gain as the signal is increasingly attenuated.

The rate or slope of this falloff is determined by the filter's order, which comes from the transfer function. The transfer function's maximum number of roots in the denominator, which are called poles, is the circuit's order. The slope of the drop off is $-20n$ dB/decade where n is the order. Higher order filters have larger slopes. For example, the slope is -20 dB/decade for a first order and -40 dB/decade for a second order filter.

For bandpass and bandstop filters, the **bandwidth** B is the difference between the high and low cutoff frequencies

$$B = f_H - f_L \tag{10.6}$$

where f_H is the high cutoff frequency and f_L is the low cutoff frequency.

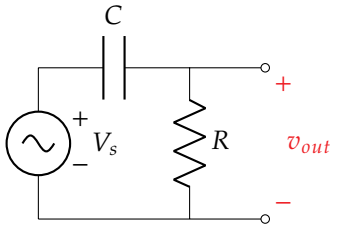
These four types of filters can be implemented using either passive or active components. Using passive components only, the simplest circuit for each of the four filters is shown in Figure 10.4. A bandstop filter can be created by combining high and low pass filters if the bandwidth is high enough that the two filters do not interfere or reduce passing frequencies too much. Similarly, a bandpass filter can be created by combining high and low pass filters; however, the two filters interfere with each other reducing the maximum output to below 0 dB. In order to limit the interference, active filters with components like op amps isolate filters. Numerous circuit designs exist for active filters.

Active and passive filters have distinct advantages and drawbacks. Active filters provide isolation between filter circuits, allow smaller component sizes (particularly capacitors), eliminate the need for expensive inductors, and can amplify gain. However, active filters require power for their active components, and op amps perform poorly with large amounts of power. They work best at frequencies well below 100 kHz, whereas passive filters perform poorly at frequencies below the audio range (300-3000 Hz).

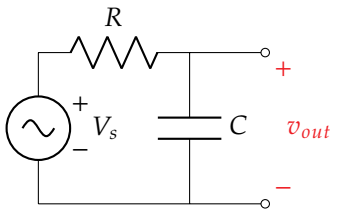
10.4 Learning Targets

AC circuits can be analyzed in the frequency domain now that you can

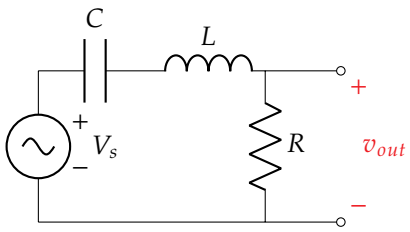
- define a transfer function both mathematically and in words.
- convert transfer function gain to units of decibels to express ratios of power, voltage, or current.
- define a bode plot and interpret bode plot graphs, mainly to determine filter type.
- describe the function of filter circuits, classify them as passive or active, and determine filter type: low-pass, high-pass, band-pass, and band-stop.
- design a filter.



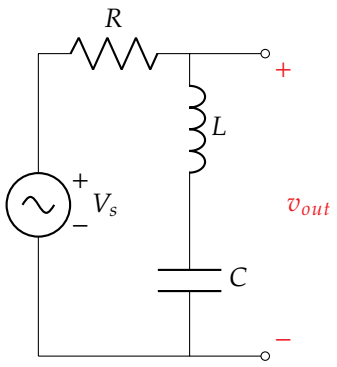
(a) High-pass Filter



(b) Low-pass Filter



(c) Bandpass Filter



(d) Bandstop Filter

Figure 10.4: Passive Filter Circuits for the Four Filter Types

Greek Letters with Pronunciations

Character	Name	Character	Name
α	alpha *AL-fuh*	ν	nu *NEW*
β	beta *BAY-tuh*	ξ, Ξ	xi *KSIGH*
γ, Γ	gamma *GAM-muh*	o	omicron *OM-uh-CRON*
δ, Δ	delta *DEL-tuh*	π, Π	pi *PIE*
ϵ	epsilon *EP-suh-lon*	ρ	rho *ROW*
ζ	zeta *ZAY-tuh*	σ, Σ	sigma *SIG-muh*
η	eta *AY-tuh*	τ	tau *TOW (as in cow)*
θ, Θ	theta *THAY-tuh*	υ, Υ	upsilon *OOP-suh-LON*
ι	iota *eye-OH-tuh*	ϕ, Φ	phi *FEE, or FI (as in hi)*
κ	kappa *KAP-uh*	χ	chi *KI (as in hi)*
λ, Λ	lambda *LAM-duh*	ψ, Ψ	psi *SIGH, or PSIGH*
μ	mu *MEW*	ω, Ω	omega *oh-MAY-guh*

Capitals shown are the ones that differ from Roman capitals.

Trig Identities Reference

$$\begin{aligned}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\sin(\omega t \pm 180\circ) &= -\sin \omega t \\
\sin(\omega t \pm 90\circ) &= \pm \cos \omega t \\
\\
\cos(A \pm B) &= \cos A \cos B \pm \sin A \sin B \\
\cos(\omega t \pm 180\circ) &= -\cos \omega t \\
\cos(\omega t \pm 90\circ) &= \mp \sin \omega t \\
\\
A \cos t + B \sin t &= C \sin(\omega t + \phi) \\
&= C \cos(\omega t - \phi) \\
&\text{where} \quad C = \sqrt{A^2 + B^2}, \phi = \tan^{-1} \frac{B}{A}
\end{aligned}$$

Imaginary number $j := \sqrt{-1}$

Euler's formula

$$e^{j\theta} = \cos\theta + j \sin\theta$$

where the real and imaginary parts are

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \qquad \sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$